THE

PHILOSOPHY OF ARITHMETIC

AS DEVELOPED FROM THE

THREE FUNDAMENTAL PROCESSES

OF

SYNTHESIS, ANALYSIS, AND COMPARISON

CONTAINING ALSO

A HISTORY OF ARITHMETIC

BY

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"The highest Science is the greatest simplicity."

LANCASTER, PA.:
NORMAL PUBLISHING COMPANY.
1880.

STEREOTYPED & PRINTED
BY
THE INQUIRER P. & P. CO.,
LANCASTER, PA

PREFACE.

PROGRESS in education is one of the most striking characteristics of this remarkable age. Never before was there so general an interest in the education of the people. The development of the intellectual resources of the nation has become an object of transcendent interest. Schools of all kinds and grades are multiplying in every section of the country; improved methods of training have been adopted; dull routine has given way to a healthy intellectual activity; instruction has become a science and teaching a profession.

This advance is reflected in, and, to a certain extent, has been pioneered by, the improvements in the methods of teaching arithmetic. Fifty years ago, arithmetic was taught as a mere collection of rules to be committed to memory and applied mechanically to the solution of problems. No reasons for an operation were given, none were required; and it was the privilege of only the favored few even to realize that there is any thought in the processes. Amidst this darkness a star arose in the East; that star was the mental arithmetic of Warren Colburn. It caught the eyes of a few of the wise men of the schools, and led them to the adoption of methods of teaching that have lifted the mind from the slavery of dull routine to the freedom of independent thought. Through the influence of this little book, arithmetic was transformed from a dry collection of mechanical processes into a subject full of life and interest. The spirit of analysis, suggested and developed in it, runs to-day like a golden thread through the whole science, giving simplicity and beauty to all its various parts.

No one who did not in his earlier years learn arithmetic by the old mechanical methods, and who has not experienced the transition to the new analytic ones, can realize the completeness of the revolution effected by this little work. But great as has been its influence, it should be remembered that it does not contain all that is essential to the science of numbers. Analysis in its mission, has done all that it was possible for it to accomplish, but it is not sufficient for the perfection of a science. There must be synthetic thought to build up, as well as analytic thought to separate and simplify. Comparison and generalization have an important work to perform in unfolding the relations of the various parts and in uniting them by the logical ties of thought, which should bind them together into an organic unity. What we now need for the perfection of the science of arithmetic and our methods of teaching it, is a more philosophical conception of its nature, and a logical relating of its parts which analysis leaves in a disconnected condition.

It is worthy of remark that arithmetic, in respect to logical symmetry and completeness, differs widely from its sister branch—geometry. The science of geometry came from the Greek mind almost as perfect as Minerva from the head of Jove. Beginning with definite ideas and self-evident truths, it traces its way, by the processes of deduction, to the profoundest theorem. For clearness of thought, closeness of reasoning, and exactness of truths, it is a model of excellence and beauty. It stands as a type of all that is best in the classical culture of the thoughtful mind of Greece. Geometry is the perfection of logic; Euclid is as classic as Homer.

The science of numbers, originating at the same time, seems to have presented less attractions or greater difficulties to the Greek mind. It is true that the great thinkers grew enthusiastic in the contemplation of numbers, and spent much time in fanciful speculations upon their properties, but this did comparatively little for the development of the science. The present system of arithmetic is mainly the product of the thought of the past three or four centuries. Developed by minds less logical than those of the old

Greeks, and growing partly out of the necessities of business, it seems not to have acquired that scientific exactness and finish which belong to the science of geometry. That it has intrinsically as logical a basis and will admit of as logical a treatment, cannot be doubted. To endeavor to exhibit the true nature of the science, show the logical relation of its parts, and thus aid in placing it upon a logical foundation beside its sister branch, geometry, is the object of the present treatise.

The work is divided into *five parts*, besides the Introduction. The Introduction contains a Logical Outline of Arithmetic, and a brief History of the science, including an account of the Origin of the Arabic system, the Origin of the Fundamental Operations, and an account of the Early Writers on the science. The facts presented have been gathered from a variety of sources, and have been carefully compared, so far as was possible, with the originals, to secure entire accuracy in the statements. The principal authorities followed are Leslie, Peacock, and De Morgan. As much is presented as it is supposed will be of interest to the teacher or general reader; any who desire more detailed information are referred to the writers mentioned.

Part First treats of the general nature of arithmetic, embracing the *Nature of Number*, the *Nature of Arithmetical Language*, and the *Nature of Arithmetical Reasoning*. The nature of Number is quite fully considered, especially in its relation to the idea of Time. Various definitions of Number are presented and examined, and the effort is made to ascertain that which may be regarded as the best for general use.

The Nature of the Language of Arithmetic is discussed upon a broader basis than usual. The true relation of Numeration to Notation, which seems to have been overlooked by many authors, and which is frequently not understood by pupils, is explained. It is shown that Numeration is merely the *oral* and Notation the *written* language of Arithmetic. The philosophy of the Arabic system of notation, the objections to the decimal scale, and the advantages of a duodecimal system of arithmetic, are discussed.

Considerable attention is given to the nature of Arithmetical Reasoning, a subject which seems not to have been very clearly understood by logicians and arithmeticians. The effort is made to put this matter upon a logical basis, and to ascertain and present the true nature of the logical processes by which the science of numbers is unfolded. The ground being almost entirely new, it is not to be supposed that the investigation is at all complete; but it is hoped that what is given may induce some one to present a more thorough development of the subject.

The fundamental idea of the work is that arithmetic has a *triune basis;* that it is founded upon and grows out of the three logical processes, *Analysis*, *Synthesis*, and *Comparison*. This is a new generalization, and is believed to be correct. It has been previously maintained that all of Arithmetic is contained in the two processes, Addition and Subtraction; and that the whole science is a logical outgrowth of these two fundamental ones. In this work it is shown that *Synthesis* and *Analysis* are mechanical operations, giving rise to some of the divisions of the science, that the mechanical processes are directed by the thought process of *Comparison*, and that this itself gives rise to a larger part of the science. The old writers held that we can only *unite* and *separate* numbers; in this work it is held that we can *unite*, *separate*, and *compare* numbers.

Proceeding with this idea, it is shown that, regarding Addition, Subtraction, Multiplication, and Division, as the fundamental operations of arithmetic, there will arise from them several other processes of a similar character, which I have called the *Derivative Processes* of Synthesis and Analysis. It is then seen that for each analytical process there should be a corresponding synthetic process. There will thus arise a new process, the opposite of *Factoring*, to which I have given the name of *Composition*. This process, it will be seen, contains several interesting cases, which correlate with the different cases of Factoring. It is of especial interest in Algebra, as may be seen in my Elementary Algebra.

Continuing this thought, it is shown that *Ratio*, *Proportion*, the

Progressions, etc., are not the outgrowth of either Synthesis or Analysis, but of the thought process—*Comparison*. Attention is called to the nature of Ratio, a new definition is suggested, and the correctness of the prevailing method of finding the ratio of two numbers, which has been questioned, is vindicated. Suggestions are also made for improvements in some of the definitions and methods of treating Ratio, Proportion, Progressions, etc. The logical character of *Percentage* is exhibited, and the simplest and most practical method of treatment suggested. Several interesting chapters are also presented upon the *Theory of Numbers*.

The subject of *Fractions* is quite fully discussed, the attempt being made to exhibit their nature and their logical relation to integers. The possible cases which may arise are considered, and a new case, called the *Relation of Fractions*, first given in one of my arithmetics, and already introduced into several other arithmetical works, is presented and explained. It is also shown that the subject of Fractions admits of *two methods of treatment*, logically distinct in idea and form, and both treatments are presented. Especial attention is given to the treatment of *Circulates*, and the most important principles concerning them are collated.

The nature of *Denominate Numbers*, which seems to have been imperfectly understood, is explained upon what is regarded as the correct basis. They are shown to be numerical expressions of *continuous quantity*, in which some artificial unit is assumed as a measure. This leads to the adoption of a new definition of Denominate Numbers, different from that which we usually find in our text-books. The origin of the measures in the various classes of Denominate Numbers is also stated, and many interesting facts concerning them are given.

While the philosophical part of the work is that which will attract the most attention among thinkers, the historical part will be quite as interesting and instructive to the majority of younger readers. In the historical part, of course, no claims to original investigation are made; but the best authorities have been consulted; and, in many cases, their very language has been used, their expression being so clear and concise that I could not hope to improve it. In thus combining with the philosophy of arithmetic its history, which in many cases aids in unfolding its philosophy, I have aimed to present a work especially valuable to *students* and the *younger teachers* of arithmetic. Such a work, I feel, would have been invaluable to me in my earlier years as a teacher.

It is proper to remark that the work was mainly written about twelve years ago. This might be regarded as an advantage; for, according to the recommendation of Horace, publication should not be hurried, but "a work *should be retained till the ninth year*." Quintilian also remarks concerning his own great work on *Oratory* that he allowed time for reconsidering his ideas, "in order that when the ardor of invention had cooled I might judge of them on a more careful re-perusal, as a mere reader." In re-perusing the manuscript I see no reason for any change of opinion, in regard to any of the ideas presented, though I am conscious that the manner of presenting several subjects might, in some respects, be improved by being re-written; but I have decided to let them stand as originally conceived and expressed, thinking that they may thus gain in freshness and vividness of conception what they may lack in elegance of style. A few of the peculiar ideas have already been presented in one or two of my text-books, and my *logical outline* of arithmetic was, by my permission, adopted in a work on mathematics by Mr. Goodrich of New Haven.

Cherishing many pleasant remembrances associated with the discussion of these ideas before my pupils in the class-room, to many of whom their publication will prove a reminder of days gone by, I commit the work, with its merits and demerits, to an indulgent public, with the hope that it may be of assistance to the younger members of the profession, and contribute somewhat towards the fuller appreciation of the interesting and beautiful science of numbers.

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January 16, 1876.

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INTRODUCTION

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THE PHILOSOPHY OF ARITHMETIC.

INTRODUCTION.

CHAPTER I.

A LOGICAL OUTLINE OF ARITHMETIC.

THE Science of Arithmetic is one of the purest products of human thought. Based upon an idea among the earliest which spring up in the human mind, and so intimately associated with its commonest experience, it became interwoven with man's simplest thought and speech, and was gradually unfolded with the development of the race. The exactness of its ideas, and the simplicity and beauty of its relations, attracted the attention of reflective minds, and made it a familiar topic of thought; and, receiving contributions from age to age, it continued to develop until it at last attained to the dignity of a science, eminent for the refinement of its principles and the certitude of its deductions.

The science was aided in its growth by the rarest minds of antiquity, and enriched by the thought of the profoundest thinkers. Over it Pythagoras mused with the deepest enthusiasm; to it Plato gave the aid of his refined speculations; and in unfolding some of its mystic truths, Aristotle employed his peerless genius. In its processes and principles shines the thought of ancient and modern mind—the subtle mind of the Hindoo, the classic mind of the Greek, the practical spirit of the Italian and English. It comes down to us adorned with the offerings of a thousand intellects, and sparkling with the gems of thought received from the profoundest minds of nearly every age.

And yet, rich as have been the contributions of the past, few of the great thinkers have endeavored to unfold its logical relations as a science, and discover and trace the philosophic thread of thought that binds together its parts into a complete and systematic whole. Unlike its sister branch geometry, which came from the Greek mind so perfect in its symmetry and classic in its logic, the science of arithmetic has been treated too much as a system of fragments, without the attempt to coördinate its parts and weave them together with the thread of logic into a complete unity. To remedy this defect is the special object of a work on the Philosophy of Arithmetic, and is the task which the author of the present work has with diffidence attempted.

Like all science, which is an organic unity of truths and principles, the science of arithmetic has its fundamental ideas, out of which arise subordinate ones, which themselves give rise to others contained in them, and all so related as to give symmetry and proportion to the whole. What are these fundamental and derivative ideas, what is the law of their evolution, what is the philosophical character of each individual process, and what is the logical thread of thought that binds them all together into an organic unity? These are the questions that meet us at the threshold of the effort to unfold a philosophy of arithmetic; they are the foundation upon which such a superstructure must be erected; and we begin the answer to these questions in the first chapter, under the head of *A Logical Outline of Arithmetic*, which exhibits the fundamental operations and divisions of the science.

To this Logical Outline the special attention of the reader is invited, as it is not only the foundation upon which the author has builded, but also the frame-work of the system. In it the science is assumed to be based upon the three processes—Synthesis, Analysis, and Comparison; general processes in which each individual process must have its root, and from which it is developed. This generalization marks a new departure in the method of regarding the science, and the relation of its parts; and shows the incorrectness of opinions around which has gathered the dust of centuries. Our first inquiry is, what is *A Logical Outline of Arithmetic?*

All numerical ideas begin with the Unit. It is the origin, the basis of arithmetic. From it, as a fundamental idea, originate all numbers and the science based upon them. Beginning, then, at the Unit, let us see how the science of arithmetic originates and is developed.

The Unit can be multiplied or divided. This gives rise to two classes of numbers, Integers and Fractions. Integers originate in a process of synthesis, Fractions in a process of analysis. Each Integer is a synthetic product derived from a combination of units; each Fraction is an analytic product derived from the division of the unit. There are, therefore, two general classes of numbers, Integers and Fractions, treated of in the science of arithmetic.

Having obtained numbers by a combination of units, we may unite two or more numbers and thus obtain a larger number by means of synthesis; or we may reverse the operation and descend to a smaller number by means of analysis. Numbers, therefore, can be united together and taken apart; they can be synthetized and analyzed; hence Synthesis and Analysis are the two fundamental operations of arithmetic. These fundamental operations give rise to others which are modifications or variations of them. Arithmetic, therefore, from its primary conception seems to consist of but two things,—to increase and to diminish numbers, to unite and to separate them. Its primary operations are Synthesis and Analysis.

To determine when and how to unite, and when and how to separate, we employ a process of reasoning called Comparison. This process compares numbers and determines their relations. Synthesis and Analysis are mechanical processes; Comparison is the thought process. Comparison directs the original processes, modifies them so as to produce from them new ones, and also gives rise to other processes not contained in the original ones. It is, in other words, by this thought process working upon the idea of number, that the original processes of Synthesis and Analysis are directed and modified, that other processes are developed from them, and that new and independent processes arise, and the science of arithmetic is developed. Comparison, therefore, in arithmetic as in geometry, is the process by which the science is constructed, or the key with which the learner unlocks its rich storehouse of interest and beauty.

Arithmetic, it is thus seen, consists fundamentally of three things; *Synthesis*, *Analysis* and *Comparison*. *Synthesis* and *Analysis* are fundamental mechanical operations, suggested in the formation of numbers; *Comparison* is the fundamental thought process which controls these operations, eliminates their potential ideas, and also gives rise to other divisions of the science growing immediately out of itself. In other words, the science of arithmetic has a triune basis; it has its roots in, and grows out of, the three processes, *Synthesis*, *Analysis*, and *Comparison*. Let us examine these processes and see the number, nature, and relations of the divisions growing out of the fundamental operations, and thus determine the logical character of the science of arithmetic.

SYNTHESIS.—A general synthesis is called *Addition*. A special case of the synthetic process of Addition, in which the numbers added are all equal, their sum receiving the name of product, is called *Multiplication*. The forming of *Composite Numbers* by a synthesis of *factors*, which may be called *Composition; Multiples*, formed by a synthesis of *particular* factors; and *Involution*, by a synthesis of *equal* factors, are all included under Multiplication. Hence, since Involution, Multiples, and Composition, are special cases of Multiplication, and Multiplication is itself a special case of Addition, the process of Addition includes all the synthetic processes to which numbers can be subjected.

ANALYSIS.—A general analysis, the reverse of Addition, is called *Subtraction.* A special case of Subtraction, in which the same number or equal numbers are successively subtracted with the object of ascertaining how many times the number subtracted is contained in another, is called *Division. Factoring* is a special case of Division in which *many* or *all* of the factors of a number are required; *Evolution* is a special case of factoring in which *one* of the several *equal* factors is required; and *Common Divisor* is a case of factoring in which some *common* factor of several numbers is required. The process of Division, therefore, includes the processes of Factoring, Common Divisor, and Evolution; and since Division is a special case of Subtraction, all of these processes are logically included under the general analytic process of Subtraction.

COMPARISON.—By comparison the general notion of relation is attained, out of which arise several distinct arithmetical processes. By comparing numbers, we perceive the relations of difference and quotient; and giving measures to these, we have *Ratio.* A comparison of *equal* ratios gives us *Proportion.* A comparison of several numbers differing by a *common* ratio gives us *Arithmetical* and *Geometrical Progression.* In comparing concrete numbers, when the unit is artificial, we perceive that they differ in regard to the value of the units, and also that we can change a number of units of one species into a number of another species of the same class; and thus we have the process called *Reduction.* In comparing abstract numbers we notice certain relations and peculiarities which, investigated, give rise to the *Properties* or principles of numbers. In comparing numbers, we may assume some number as a basis of reference and develop their relations in regard to this basis;—when this basis is *a hundred*, we have the process called *Percentage.*

Thus we obtain a complete outline of the science of numbers, and perceive more clearly the logical relations of the divisions of the science. Arithmetic is conceived as based upon the two fundamental operations, *synthesis* and *analysis*, these operations being controlled by *comparison*, which develops new processes from these and also from itself. The whole science of Pure Arithmetic is the outgrowth of this triune basis, Synthesis, Analysis, and Comparison. The rest of arithmetic consists of the solution of problems, either real or theoretic, and may be included under the head of Applications of Arithmetic.

This conception of the subject is new and important. It has been heretofore held that addition and subtraction comprehended the entire science of arithmetic; that all other processes are contained in them, and are an outgrowth from them. This is a fallacy, which, among other things, has led logicians to the absurd conclusion that there is no reasoning in arithmetic. Assuming that there is no reasoning in the primary processes of synthesis and analysis, and that these primary processes contain the entire science, they naturally conclude that there is no reasoning in the science itself. The analysis of the subject here given dispels this error and exhibits the subject in its true light. Synthesis and Analysis are seen to be the primary mechanical processes; Comparison, the thought process, touches them with her wand of magic, and they germinate and bring forth other processes, having their roots in these primary ones. Comparison also becomes the foundation of processes distinct from those of synthesis and analysis, processes which cannot be conceived as growing out of synthesis and analysis, but which have their root in the thought process of the science—in Comparison.

This outline of the science grows out of the pure idea of number, independent of the language of arithmetic. These fundamental processes are modified by the method of notation employed to express numbers. With the Roman or Greek methods of notation, the methods of operation would not be the same as with the Arabic system. The method of "carrying one for every ten," of "borrowing" in subtracting, the peculiar methods of multiplying and dividing, grow out of the Arabic system of notation. A portion of the treatment of common and decimal fractions arises from the notation adopted, and the principles and processes of repetends originate in the same manner. The methods of extracting square and cube root would be different if we employed a different method of expressing numbers. It is thus seen that the fundamental divisions of arithmetic arise from the pure idea of number, that the processes in these divisions are modified by the method of notation adopted, and also that some of the principles and processes of the science grow out of this notation. It may be remarked, also, that the power of arithmetic as a calculus depends upon the beautiful and ingenious system of notation adopted to express numbers.

It is believed that the above view of arithmetic must tend to simplify the subject, and that much clearer notions of the science will be attained when these philosophical relations are apprehended. A general view of the subject is presented by the following analytical outline:

Logical Outline of Arithmetic.

- I. Synthesis.
  - Addition.
  - Multiplication.
    - Composition.
    - Common Multiple.
    - Involution.
- II. Analysis.
  - Subtraction.
  - Division.
    - Factoring.
    - Common Divisor.
    - Evolution.
- III. Comparison.
  1. Ratio.
  2. Proportion.
  3. Progression.
  4. Reduction.
  5. Percentage.
  6. Properties of Numbers.

LOGICAL OUTLINE OF ARITHMETIC.

- I. SYNTHESIS.
  - ADDITION.
  - MULTIPLICATION.
    - COMPOSITION.
      - COMMON MULTIPLE.
      - INVOLUTION.
- II. ANALYSIS.
  - SUBTRACTION.
  - DIVISION.
    - FACTORING.
      - COMMON DIVISOR.
      - EVOLUTION.
- III. COMPARISON.
  1. RATIO.
  2. PROPORTION.
  3. PROGRESSION.
  4. REDUCTION.
  5. PERCENTAGE.
  6. PROPERTIES OF NUMBERS.

CHAPTER II.

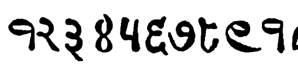
ORIGIN AND DEVELOPMENT OF ARITHMETIC.

THE science of Arithmetic is coeval with the race. Every people, no matter how uncivilized, must have possessed some ideas of numbers, and employed them in their transactions with each other. These ideas would be multiplied, and the methods of operation founded upon them gradually extended and improved as the nation advanced in civilization and intelligence. The history of Arithmetic is, therefore, inseparably connected with the history of civilization and the race. The origin of its elementary processes must, of necessity, be involved in obscurity and uncertainty. History can speak positively only of some of the higher and more recent developments of the science.

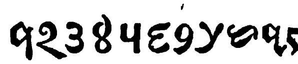
In presenting what is known concerning the history of arithmetic, we shall consider three things: the origin of our present system of arithmetic; the origin of the fundamental operations; and the early writers on the science. Other historical facts will be mentioned in connection with the particular subjects to which they belong. One of the most interesting inquiries is that which relates to the origin of the system of arithmetic now generally adopted, which we shall consider in the present chapter.

The origin of our present system of Arithmetic has been attributed to various nations. The Greeks, the Chaldeans, the Phœnicians, the Egyptians, and the Hebrews, have all been claimed as its inventors. Gatterer, an eminent German historian, tells us that ciphers were primordial letters, invented by Taaut, or Thaut, with which the ancient Egyptians and Phœnicians were well acquainted; and that they gradually became known to all Oriental nations, among whom they were preserved until the victorious arms of the Mussulman penetrated into India, and brought back those precious monuments of genius. This, however, is very improbable, since there are no characters upon the Egyptian monuments that bear any resemblance to those of our denary system. Like the Chinese and some others, they are abridged representations of objects rather than arbitrary signs.

It is now generally believed that this system originated among the Hindoos. The people of Hindostan, and nations communicating with them, have, for many centuries, been acquainted with the denary system, and the most careful investigations point to the Hindoos as the inventors of the system. Their sacred books, which have been in the hands of the priesthood for centuries, contain the numeral characters, quite similar in form to those now in use. We give in the margin, copied from Leslie, the Sanskrit digits, in what are called the *Devanagari* character.



From these the common Hindoo digits, also presented in the margin, are supposed to have been formed, with only a slight alteration of form. The Burman figures are evidently of the same origin, but have a thin wiry body, being generally written on the palmyra leaf with the point of a needle.



Among the sacred writings of the Hindoos, there is preserved a treatise on Arithmetic and Mensuration, written in the Sanskrit language, called Lilawati. This was regarded as of such inestimable value as to be ascribed by them to the immediate inspiration of Heaven. After an introductory preamble and colloquy of the gods, it begins with the expression of numbers by nine digits and the cipher, or small *o*. The characters are similar to those in present use, and the method of notation is the same. It contains the common rules of arithmetic, and the extraction of the square root as far as two places. The exam ples are generally very easy, scarcely forming any part of the text, and are written in the margin with red ink. This work is very old, and proves that the Hindoos have possessed this system for many centuries. Their knowledge of the science, however, is quite limited. They have no idea of the decimal scale descending, and their management of fractions is tedious and embarrassed. But notwithstanding the limited character of their knowledge, we are unable to trace the origin of the science to any earlier source, and to them must be ascribed the honor of its invention. They disclaim this honor, however, referring it to Divinity, maintaining that the invention of the nine figures and the device of place, is to be ascribed to the beneficence of the Creator of the universe.

It has been thought strange that a people so little celebrated in science and the history of thought should have originated so important a system as the digital method of notation, while other nations, much more eminent in science and philosophy, seemed to have been unequal to the task. Leslie says, "If the exuberant fancy of the Greeks led them far beyond the denary notation, it seems probable that the feebler genius of the Hindoos might just reach the desirable point, without diverging into an excursive flight." A more intimate knowledge of the Hindoos, however, shows that they are remarkable for ingenuity and subtlety of thought; and this invention is therefore not to be regarded as the product of a feeble mind, but as one of the highest triumphs of inventive genius known in the history of scientific investigation.

It was for a long time supposed that the present system of arithmetic was due to the Arabians. The characters in general use were named Arabic characters, and the method of expressing numbers was called the Arabic method of notation. This is also indicated by the two words "cipher" and "zero." Cipher is the Arabic "sifrun," which means empty, a translation of the Sanskrit name of the naught, "sunya." The same character, the naught, is called "zephiro" in Italian, which has, by rapid pronunciation, been changed to "zero." This form occurs as early as 1491, in a work of Philip Calandri on Arithmetic, published in Florence. The influence exerted by the Arabs in the introduction of the system into Europe led to the belief that it originated with them, and caused it to be called the Arabic system.

The Arabs, however, it is now positively known, were not the authors of the system. It appears that they were not acquainted with the denary characters before the twelfth or thirteenth century. They cultivated mathematics with ardor, but seldom aspired to original efforts, contenting themselves generally with copying their Grecian masters. It seems probable, from all the information we have upon the subject, that they did not adopt the denary system until after an easy communication was opened with Hindostan. They might have derived it through the Persians, who were conquered by them in 636, and whose religion and at least one of their dialects they had adopted; and who, like the Arabs themselves, were distinguished for a love of science and a spirit of conquest. The Arabic numerals resemble the Persian, which are now current over India, and are there esteemed the fashionable characters. It should be stated also that the Arabs do not claim the invention of these figures, but universally ascribe it to the Indians, and there is now very little doubt that the Brahmins were the original inventors of the numerical symbols and the system of denary arithmetic.

It is an interesting fact that Boëthius, in his work *De Geometria*, informs us that the disciples of Pythagoras used in their calculations *nine peculiar figures*, while others used the letters of the alphabet. It is probable that this philosopher, who had traveled considerably, had obtained this knowledge in Hindostan; and communicating it as a secret to his disciples, caused it to remain sterile in their hands.

It still remains a question when and by whom these characters were originally introduced into Europe. By some, the period is fixed at or near the beginning of the 11th century; others suppose it to have been at least 250 years later. The question is one of much difficulty, especially since this introduction was anterior to the invention of printing. The general belief is that the system was introduced into Europe by the Arabs. While the Christian world was enveloped in ignorance, the Arabs were cultivating, with great success, the learning and literature of Greece. Though not highly gifted with creative powers of mind by which they made many valuable additions to what they thus acquired, they are to be honored because they "preserved and fanned the holy fire." Their efforts at conquest had been crowned with brilliant success Spain had yielded to their sway, and the Moors had become celebrated throughout Europe for the splendor of their institutions and the proficiency of their scholars.

Disgusted with the trifling of their own schools, energetic and aspiring young men from England and France repaired to Spain to learn philosophy from the accomplished Moors. There they studied arithmetic, geometry, and astronomy, and made themselves familiar with the Arabic method of notation and calculation. On their return they brought the characters and methods of the Arabic arithmetic with them and introduced them to the scholars of Northern Europe, and thus in time they gradually displaced the Roman system.

Efforts have been made to ascertain what persons were most conspicuous in the introduction of the Arabic characters and in leading the people of Europe to make the transition from the Roman to the Arabic system of calculation. Among the most celebrated of these "pilgrims of science" was an obscure monk of Auvergne, named Gerbert. Possessing an ardent love for science, he applied himself with great energy to the study of their mathematics, consisting of arithmetic, geometry, and astronomy. Returning to his native country, after the completion of his studies, he became widely celebrated for his genius and learning; and, as was natural with a people enveloped in ecclesiastical ignorance and superstition, who remembered that he had pursued his studies among infidels, there were not wanting those who represented him as being a magician and in league with the Evil One. But so great were his genius and influence, that he triumphed over the accusations of malice, and rose eventually to the Papal chair, which position he filled under the title of Sylvester II. Gerbert died in the year 1003, leaving a legacy of valuable learning to the world. He wrote extensively upon the subjects he had studied in Spain. His treatises upon arithmetic and geometry were valuable, presenting many rules for abbreviating the operations then in common use. In some manuscript copies of his works, the numbers are expressed in the denary characters, and it is therefore claimed that to him belongs the honor of their introduction into Europe. Others suppose, however, that these characters crept in through the license of transcribers, and maintain that we are not warranted in concluding that Gerbert had the merit of introducing the Arabic symbols to Northern Europe.

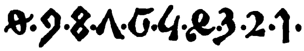
Another candidate for the honor of the introduction is Roger Bacon. This claim is based upon an old almanac preserved in the Bodleian Library, at Oxford, which contains numerals in their earliest form, and of which, in the general spirit of assigning to Bacon all the great inventions and discoveries of the times whose origin is shrouded with mystery or uncertainty, it was supposed that he was the author, and hence the introducer of the denary characters. The claim, however, instantly ceases when it is remembered that the work is dated 1292, the same year that Bacon died, and that it was calculated for the meridian of Toledo. It is very probable that it was written in France and imported into England. John of Halifax, a cotemporary of Bacon, wrote a work about this time which, in several manuscript copies, contains the denary numerals; and the honor, therefore, has been also ascribed to him. But it is probable that these ciphers, as they were called, were introduced, not by the author, but by subsequent copyists.

Leslie thinks that the modern system of arithmetic, as well as the higher art of algebra, was first brought into Europe by Leonardo Bonacci, of Pisa, a wealthy merchant who traded on the coast of Africa and the various ports of the Levant Tempted by commercial speculation to visit those countries, he was induced by a love of knowledge to study thoroughly the science of calculation among the Arabians. On his return to Italy in 1202, he composed a treatise on arithmetic which he enlarged in 1228. His manuscript lay more than two centuries neglected when Lucas Pacioli, or Di Borgo, instructed chiefly by its perusal, published successively between the years 1470 and 1494, the earliest and most extensive printed treatise on arithmetic and algebra.

Denary numerals, it is known, were first used by the astronomers of France and Germany in composing calendars which were sent to the various religious houses. By this means the characters were generally diffused throughout Europe. Gerard Vossius, a learned man of the 16th century, supposes this to have been about the year 1250. Du Cange, by some regarded as a more reliable authority, maintains that it could not have been previous to the 14th century, as ciphers were not known before that time. Father Mabillon, a man of great research, assures us that he met them in dates previous to the year 1400. Kircher, generally known as "Father Kircher," a Jesuit of vast acquirements, refers the introduction of the numerals to the astronomical tables published by Alphonso, King of Castile, in 1252. It is suspected, however, that in the original work the Roman or Saxon characters were employed. Two letters from that celebrated prince to Edward I., which are still preserved in the Tower of London, bearing dates 1272 and 1278, are written in the ancient characters. This latter fact, however, is not conclusive, since in writing a letter he would naturally employ the characters in common use.

There is an almanac preserved in the library of Bennet College, Cambridge, containing a table of eclipses for the period from 1330 to 1348. This almanac contains a brief explanation of the use of numerals and the principles of the denary notation. From this we may infer, that the use of the denary notation was, at this date, very imperfectly understood.

A little tract in the German language entitled *De Algorismo*, bearing the date 1390, explains with great brevity the digital notation and the elementary rules of Arithmetic. At the end of a short missal, similar directions are given in verse, which from the form of the writing seem to belong to the same period. The characters, of which these in the margin are an exact fac simile, are in both manuscripts written from right to left, the order which the Arabians would naturally follow.



The great Italian poet, Petrarch, has the honor of leaving us one of the oldest authentic dates in the numeral characters. The date is 1375, written upon a copy of St. Augustine. The college accounts in the English universities were generally kept in the Roman numerals until the beginning of the sixteenth century. The Arabic characters were not used in the parish-registers before 1600. The oldest date met with in Scotland is that of 1490, which occurs in the rent-roll of the Diocese of St. Andrew's.

It must have been about this period that a knowledge of the denary method of notation began to be spread over Europe. Eminent scholars may have been familiar with it some few years before, but at this time it began to be generally known and practiced among learned men.

The forms of several of the figures have undergone considerable change since their first introduction into Europe. In the oldest manuscripts. the figures, 4, 5, and 7 are most unlike the present characters. The 4 consists of a loop with the ends pointing down; the 5 has some likeness to the figure 9; and the 7 is simply an inverted V. In the dates used by Caxton in the year 1480, the 4 has assumed its present shape, but the

5 and 7 are still very unlike the same characters of to-day. No reason is assigned for these changes; they appear to have been gradual, and the result of chance rather than intention. It may also be stated that no attempt is made to attach any especial significance to the forms of the numerals; if there be a symbolism hidden in their form, it has yet to be unveiled.

This explanation of the introduction of the present system of arithmetic into Europe has been generally received as the true one; recent investigations, however, have thrown some doubt upon it. It is supposed that these numerical symbols had found their way into Europe before the invasion of Spain by the Mohammedans. A work, through which the Arabs in the ninth century became initiated into the science of Indian ciphering and arithmetic, is extant. This work was founded on treatises brought from India to Bagdad in 773, and was translated again into Latin during the Middle Ages. It is also asserted that the figures used in the principal countries of Europe during the Middle Ages, and, with some modifications, at the present day, differ considerably from the figures used in the East, and approach very nearly to those used by the Arabs in Africa and Spain. This would seem to indicate that the Arabs did not bring their figures, which they had learned in the East, with them into Spain. Again, it is supposed that the Neo-Pythagoreans, who were probably the first teachers of ciphering among the Greeks and Romans, became acquainted with the Indian figures, and adapted them to the Pythagorean Abacus, and that Boethius, or his continuator, made these figures generally known in Europe by means of mathematical hand-books, and that thus, long before the time of Gerbert, who, Müller says, probably never went to Spain, these same figures had found their way from Alexandria into European schools and monasteries.

One of the latest writers upon this subject, M. Woepcke, an excellent Arabian scholar and mathematician, says that the Arabs have two sets of figures, one used chiefly in the East, which he calls "Oriental;" another used in Africa and Spain, and there called "Gobar." "Gobar" means "dust," and these figures were so called because, as the Arabs say, they were first introduced by an Indian who used a table covered with fine dust, for the purpose of ciphering. Both sets of figures are called Indian by the Arabs. The Gobar figures are said to be modifications of the initial letters of the Sanskrit numerals, and could not have been derived from India much after the third or fourth century. They were probably adopted by the Neo-Pythagoreans, and introduced into Italy and the Roman provinces, Gaul and Spain, as early as the sixth century; so that the Mohammedans, when arriving in Spain in the eighth century, found these figures there already established. The Arabs, when starting on their career of conquest, it is said, were hardly able to read or write; they were certainly ignorant of ciphering, and could not, therefore, be considered as the original propagators of the so-called Arabic figures.

We can well understand, therefore, that the Arabs, on arriving in Spain, without any considerable knowledge of arithmetic, should have adopted there, as they did in Greece and Egypt, the figures which they found in use, and which had traveled thither from the Neo-Pythagorean schools of Alexandria, and originally from India; and likewise that when, in the ninth and tenth centuries, the new Arabic treatises on arithmetic arrived in Spain from the East, they should have adopted the more perfect system of ciphering carried on without the Abacus, and rendering, in fact, its columns unnecessary by the judicious employment of the naught. But, while dropping the Abacus, there was no necessity for their discontinuing or changing the figures to which the Arabs, as well as the Spaniards, had been accustomed for centuries; and hence we find that the Arabic figures were retained in Spain, only adapted to the purpose of the new Indian arithmetic by a more general use of the naught. The naught was known in the Neo-Pythagorean schools; but with the columns of the Abacus it was superfluous, while, with the introduction of ciphering in fine powder, and without columns, its use naturally became very extensive. As the system of ciphering in fine powder was called "Indian," the Gobar figures, too, were frequently spoken of under the same name, and thus the Arabs in Spain brought themselves to believe that they had received both their new arithmetic and their figures from India; the truth being, according to M. Woepcke, that they had received their arithmetic from India directly, while their figures had come to them from India indirectly, through the mediation of the Neo-Pythagorean schools.

M. Woepcke thinks that the Indian figures reached Europe through two channels; one passing through Egypt about the third century; another passing through Bagdad in the eighth century, and following the track of the victorious Islam. The first carried the earlier forms of the Indian figures from Alexandria to Rome, and as far as Spain; the second carried the later forms from Bagdad to the principal countries conquered by the Khalifs, with the exception of those where the earlier or Gobar characters had already taken firm root. He regards our figures as modifications of the early Neo-Pythagorean or Gobar form, and admits their presence in Europe long before the science and literature of the Arabs in Spain could have reacted on the seats of classical literature. The only change produced in the ciphering of Europe by the Arabs was, he supposes, the suppression of the Abacus, and the more extended use of the cipher. He thinks our figures are still the Gobar figures, written in a more cursive manner by the Arabs of Spain; and that those who, in the twelfth century, went to Spain to study Arabic and mathematics, learned there the same numerals which Boethius, or his continuator, taught in Italy in the sixth century. In the MSS. of the thirteenth and fourteenth centuries, the figures vary considerably in different parts of Europe, but they are at last fixed and rendered uniform by the introduction of printing

CHAPTER II

EARLY WRITERS ON ARITHMETIC.

THE earliest writers upon the science of arithmetic of whom we have any definite knowledge, are the mathematicians of Greece. They cultivated the science to some considerable extent, and early distinguished between the theory and practice of arithmetic. Considered in relation to its principles, they called it Logistic; and as a collection of rules, the Art of Arithmetic. The science with them was speculative, abounding with fanciful analogies.

It is a matter of surprise that the Greeks, so intellectual and cultivated, did not invent a simple and convenient method of arithmetical notation. Their system was complicated and inconvenient, rendering calculation with large numbers difficult and tedious. The land of Plato and Aristotle, it would seem, should have been equal to the task of inventing the simple, and apparently evident system of notation now in general use. A commentator, in allusion to this, says: "The ingenuity and varied resources of the Greeks were the main causes which diverted them from discovering our simple denary system. Their exuberant fancy led them beyond the denary scale; the feeble genius of the Hindoos might just reach it without mounting into an excursive flight." More recent investigations have given us clearer ideas of the philosophical character of the Hindoo mind, and serve to modify this opinion.

Pythagoras, an eminent geometer, who lived about 600 B. C., was one of the earliest writers upon mathematics. He brought from the East a passion for the mysterious properties of numbers, under the veil of which he probably concealed some of his secret and esoteric doctrines. He regarded numbers as of Divine origin—the fountain of existence—the model and archetype of things—the essence of the universe. He divided them into classes, to each of which were assigned distinct and peculiar properties. They were Prime and Composite; Perfect and Imperfect; Redundant and Defective; Plane and Solid; Triangular, Square, Cubical, Pyramidal. Even numbers were regarded as feminine; odd numbers were masculine, partaking of celestial natures.

Euclid was the first writer upon arithmetic whose works have come down to us. His treatise is contained in the 7th, 8th, 9th and 10th books of Euclid's Elements, in which he treats of proportion and of prime and composite numbers. These books are not included in the common editions of Euclid, but are found in an edition by the celebrated Dr. Barrow. It is supposed that Euclid was quite largely indebted to Thales and Pythagoras for his knowledge of the subject, though he undoubtedly added much to the science himself. His school at Alexandria was highly celebrated, being attended by the Egyptian monarch, Ptolemy Lagus. It was this pupil to whom Euclid, upon being asked if there was an easier method of learning mathematics, replied, "There is no royal road to geometry."

Archimedes, born 291 B. C., was an eminent mathematician, and made discoveries in the sciences of geometry and natural philosophy, among which are the ratio of the cylinder to the inscribed sphere and cone, the method of determining the specific gravity of bodies, etc. He no doubt added much to arithmetical science also; but in the few fragments of his writings which have come down to us, we find nothing upon the subject.

Eratosthenes, who flourished about 200 years before Christ, invented the method of determining prime numbers, known as Eratosthenes' sieve. Nicomachus, who is supposed to have lived near the Christian era, wrote a work upon the distinctions of numbers, dividing them into plane, solid, triangular, pyramidal, etc.

Diophantus, a Greek mathematician of Alexandria, who lived about the middle of the 5th century, composed thirteen books upon the subject of arithmetic, only six of which have come down to us. He is also celebrated as being the first writer upon algebra, which he applied to the investigation of the properties of numbers. He invented the method of mathematical investigation known as the Diophantine Analysis.

Boethius, the next writer of eminence, lived about the begining of the 6th century. His work, it is said, was, in the main, a copy of Nicomachus. The arithmetic of Boethius was the classical work of the Middle Ages, and became the model of many subsequent writers, even down to the fifteenth century. It was entirely theoretical, treating of the properties of numbers, particularly their ratios, and gave no rules of calculation; and we have no means of telling whether the philosophers of this school reckoned on their fingers, or used an abacus. In the manuscript editions of this work current during the 11th century, in which there is a description of the *Mensa Pythagorea*, also called the abacus, mention is made of nine figures which are ascribed to the Pythagoreans or Neo-Pythagoreans. This passage is by some considered spurious, and ascribed to a continuator of Boethius.

The oldest text-book on arithmetic employing the Arabic or Indian figures and the decimal system, is undoubtedly that of Avicenna, an Arabian physician, who lived in Bokhara about A. D. 1000. The work was found in manuscript in the library at Cairo, Egypt; and contains, besides the rules for addition, subtraction, multiplication, and division, many peculiar properties of numbers.

Lucas di Borgo, called also Lucas Pacioli, an Italian monk, was, according to Dr. Peacock, the author of the first printed treatise upon arithmetic. His great work called *Summa di Arithmetica*, was published in 1484. It is said to be the first European text-book which made use of the Arabic characters. De Morgan, however, maintains that Di Borgo's work did not appear until 1494, and that it was preceded by the works of Calandri and Peter Borgo. He says it was undoubtedly the first work printed on algebra, and probably the first on book-keeping.

Philip Calandri published a work on arithmetic at Florence in 1491. It begins with a picture of Pythagoras teaching, headed, "Pictagoras Arithmetrice introductor." His notion of division is curious. When he divides by 8, he calls the divisor 7, demanding, as it were, that quotient which, with seven more like itself, will make the dividend. He describes the rules for fractions, and gives some geometrical and other applications.

John Huswirt, in 1501, published, at Cologne, a short treatise on the Arabic system, apparently one of the earliest printed in the German language. The rules are verified by casting out nines.

Jacob Kobel, in 1514, published, at Augsburg, a work on arithmetic. The Arabic numerals are explained, but not used. The computation was by counters and Roman numerals. In the frontispiece is a cut representing the mistress settling accounts with her maid-servant by an abacus with counters.

Gaspar Lax published, at Paris, in 1515, a diffuse and extensive work on arithmetic, in small black letter. It treats only of the simple properties of numbers, and the apparent difficulty of dealing with numbers is surprising. It contains upwards of 250 pages, filled with propositions on the simplest properties of numbers, and not a number so large as 100 is given in illustration.

John Schoner edited a work on arithmetic in 1534, which he attributes to Regiomontanus. It consists of a series of demonstrated properties of numbers connected with the Arabic notation, involving not only the common rules of computation, but also such principles as that the number of figures in the cube cannot exceed three times the number in the root.

Jerome Cardan published, at Milan, in 1539, a work entitled *Practica Arithmetica.* It shows, as might have been expected from an Italian of that age, more power of computation than the French and German writers. It contains a chapter on the mystic properties of numbers, one use of which is in foretelling future events. These are mostly the numbers mentioned in the Old and New Testaments, but not altogether. In another treatise, Cardan expresses his opinion that it was Leonard of Pisa who first introduced the Arabic numbers into Europe.

Robert Recorde published his celebrated work on arithmetic about 1540. It was originally dedicated to Edward VI. The work was subsequently revised and enlarged by John Dee, and published in 1573, restoring the original dedication, which had been omitted in the edition prepared during the reign of Mary. This work was subsequently revised by Mellis, who added a third part on practice and other things, and also by Hartwell. The last edition known is by Edward Hatton, 1699, which contains an additional book called, "Decimals made easie." It is said to contain quite a number of the principles and problems of modern text-books. Recorde introduced the sign of equality (=) in a work on algebra, published in 1557. This work was called by the singular title, "Whetstone of Wit," in which he gives his reason for the symbol in the following quaint language: "And to avoid the tedious repetition of these words, I will settle, as I doe often in worke use, a pair of parallel or Gemowe lines of one length, thus, =, because noe 2 thynges can be more equalle."

Michael Stifel published, at Nuremberg, in 1544, his celebrated work entitled *Arithmetica Integra.* The first two books are on the properties of numbers, on surds and incommensurables, learnedly treated, and with a full knowledge of what Euclid had done on the subject. The third book is on algebra, and passes for the introduction of algebra into Germany. Stifel, in his preface, acknowledges his obligations to Adam Risen, and professes to have taken all his examples from Christopher Rudolph. Stifel is believed to have been the inventor of the symbols + and — to denote addition and subtraction. He introduced also the symbol of evolution, $\surd$, originally *r*, the initial of *radix* or root.

Nicolas Tartaglia, an eminent Italian mathematician, published a complete work on arithmetic in 1556. De Morgan says, "Of this enormous book I may say, as of that of Pacioli, that it wants a volume to describe it." It consists of two books, the first containing the application of arithmetic to common life, the second the foundation of the principles of algebra.

H. Baker published, in London, 1583, a work entitled "The Well-spring of Sciences. Which teacheth the perfect worke and practise of Arithmeticke." It is one of the books which break the fall from the "Grounde of Artes," to the commercial arithmetics of the next century. There are some short rules for particular cases, and great attention to the rule of practice. Among the peculiarities of the book is a notion, apparently, that none but fractions should deal with fractions; for Baker will not double $\frac{3}{8}$, for instance, by multiplying by 2, but only by dividing by $\frac{1}{2}$.

Simon Stevinus published, at Leyden, in 1585, a work which was edited by Albert Girard in 1634. This work is characterized by originality, accompanied by a great want of the respect for authority which prevailed in his time. For example, great names had made the point in geometry to correspond with the unit in arithmetic. Stevinus tells them that 0, and not 1, is the representative of the point. "And those who cannot see this," he adds, "may the Author of nature have pity upon their unfortunate eyes; for the fault is not in the thing, but in the sight which we are not able to give them." A portion of this work contains, "Les Tables d' Interest" and "La Disme." The *Disme* contains the first announcement of the use of *decimal fractions*, and De Morgan thinks that the table of compound interest suggested decimal fractions.

John Mellis, in 1588, at London, published, "A briefe instruction and maner hovv to keepe bookes of Accompts after the order of Debitor and Creditor," etc. This is the earliest English work on book-keeping by double entry, which has ever been produced. At the end of the book-keeping is a short treatise on arithmetic. Mellis says: "Truely, I am but the renuer and reviver of an auncient old copie, printed here in London the 14 of August, 1543. Then collected, published, made and set forth by one Hugh Oldcastle, Scholemaster, who, as appeareth by his treatise then taught Arithmetike and this booke, in Saint Ollaves parish in Marke Lane."

In 1596, a work entitled, "The Pathway to Knowledge," was published in London, which was a translation from the Dutch, by W. P. The translator gives the following verses, of which he is supposed to be the author:

Thirtie daies hath September, Aprill, June, and November,
Februarie, eight and twentie alone; all the rest thirtie and one.

Mr. Davies, in his Key to Hutton's Course, quotes the following from a manuscript of the date of 1570, or near it:

Multiplication is mie vexation,
And Division is quite as bad,
The Golden Rule is mie stumbling stule,
And Practice drives me mad.

Cataldi published a work on the square root of numbers at Bologna, in 1613. The rule for the square root is exhibited in the modern form, and he shows himself a most intrepid calculator. The greatest novelty of the work is the introduction of *continued fractions*, then, it seems, for the first time presented to the world. He reduces the square roots of even numbers to continued fractions, and then uses these fractions in approximation, but without the aid of the modern rule which derives each approximation from the preceding two.

Richard Witt, in 1613, published a work containing "Arithmetical questions" on annuities, rents, etc., "briefly resolved by means of certain Breviats." These Breviats are tables; and this work is said to be the first English book containing tables of compound interest. Decimal fractions are really used. The tables being constructed for ten million pounds, seven figures have to be cut off; and the reduction to shillings and pence, with a *temporary* decimal separation, is introduced when wanted. The decimal separator used is a vertical line; and the tables are expressly stated to consist of *numerators*, with 100... for a denominator.

John Napier's treatise on arithmetic was published at Edinburgh in 1617. This was a posthumous work. It contains a description of Napier's rods with applications. It is remarkable because it expressly attributes the use of decimal fractions to Stevinus It also states that Napier invented the decimal point. De Morgan says this is not correct, since 1993.273 is written $19932'7''3'''$.

Robert Fludd, in 1617 and 1619, published a work on mathematics at Oppenheim. It contains two dedications, the first, signed *Ego, homo,* to his Creator; the second, on the opposite side of the leaf, to James I. of England, signed Robert Fludd. The first volume contains a treatise on arithmetic and algebra. The arithmetic is rich in the description of numbers, the Boethian divisions of ratios, the musical system, and all that has any connection with the numerical mysteries of the sixteenth century. The algebra contains only four rules, referring for equations, etc., to Stifel and Recorde. The signs of addition and subtraction are P and M with strokes drawn through them. The second volume is strong upon the hidden theological force of numbers.

Albert Girard published a treatise on algebra at Amsterdam in 1629, which contains a slight treatise on arithmetic. The arithmetic contains no examples in division by more than one figure. On one occasion the *decimal point* is used. Girard is said to have introduced the *parenthesis* in place of the *vinculum*, which had been used by Recorde. Wm. Oughtred's *Clavis Mathematica*, a work on arithmetic and algebra of great celebrity, was first published in 1631. It retains the old or *scratch* method of division which, Dr. Peacock observes, lasted nearly to the end of the seventeenth century. He does not use the decimal point, but writes 12.3456 thus: 12|3456. The symbol for multiplication, ×, St. Andrew's cross, was introduced by Oughtred. He is also said to have first employed the symbol : : to denote the equality of ratios.

Nicholas Hunt published, in 1633, "The Hand-Maid to Arithmetick refined." The book is full on weights and measures, and commercial matters generally. It does not treat of decimal fractions, however. The author calls "decimall Arithmeticke," a division of a pound into 10 primes of two shillings each; each shilling into *six* primes of two pence each. It expresses the rules in verse, of which the following is an example:

> Adde thou upright, reserving every tenne,
> And write the digits downe all with thy pen
>
> Subtract the lesser from the great, noting the rest,
> Or ten to borrow you are ever prest.
> To pay what borrowed was think it no paine,
> But honesty redounding to your gaine.

Peter Herigone, in 1634, published at Paris a work entitled "Cursus Mathematici tomus secundus." It introduces the decimal fractions of Stevinus, having a chapter "des nombres de la dixme." The mark of the decimal is made by marking the place in which the last figure comes. Thus when 137 livres 16 sous is to be taken for 23 years 7 months, the product of 1378′ and 23583‴ is found to be 32497374⁗, or 3249 liv., 14 sous, 8 deniers.

William Webster published, in 1634, tables for simple and compound interest. This work treats decimal arithmetic as a thing known. No decimal point is recognized, only a partition line to be used on occasion. It contains the first head-rule for turning a decimal fraction of a pound into shillings, pence, and farthings. Many other interesting details will be found in De Morgan's *Arithmetical Books*, from which much of this chapter has been drawn.

CHAPTER IV.

ORIGIN OF ARITHMETICAL PROCESSES.

ONE of the most interesting points connected with the history of arithmetic, would be a full and complete account of the genesis of the different divisions and processes of the science. This, however, is impossible. The origin of the elementary or fundamental processes dates back before the invention of printing, and can never be determined. Some of the principal facts, however, upon this point, in addition to those already given, will be stated.

ARITHMETICAL LANGUAGE.—The notation of the nine digits and zero, upon which the science of arithmetic is based and developed, originated, as we have already shown, among the Hindoos, who, however, do not claim to have invented it, but regard it as a gift of Deity, which is the best proof of its possessing an antiquity antecedent to all existing records. The first Arabian author who wrote upon algebra and the Indian mode of computation is stated, with the common consent of Arabic authors, to have been Mohammed ben Musa, who flourished about the end of the 9th century; an author who is celebrated as having made known to his countrymen other parts of Hindoo science, to which he is said to have been very partial. Before the end of the 10th century, these figures, which are called *Hindasi*, from their origin, were in general use throughout Arabia. The same testimony is repeated in almost every subsequent author on arithmetic or algebra, and is completely confirmed by their writing these figures from left to right, after the manner of the Hindoos, but which is directly contrary to the order of their own writing. The use of this notation became general among Arabic writers on astronomy, as well as arithmetic and algebra, about the middle of the 10th century. We find it in the works of the astronomer, Ebn Younis, who died in the year 1008; and it is found likewise in all subsequent astronomical tables. From the Arabs, who, in the 11th century, held possession of the southern provinces of Spain, and had established a flourishing kingdom, in which the sciences were cultivated with great zeal and success, the knowledge was communicated to the Spaniards and other nations of Europe.

The Italians, from an early period, adopted the method of distributing the digits of a number into groups or periods of six, and consequently proceeding by millions. This is the method of numeration given by Lucas di Borgo, 1494. The method of reckoning by three places, as used in this country and on the Continent, seems to have originated with the Spanish. In a work on arithmetic by Juan de Ortega, 1536, we find the following method of numeration; 10, dezena; 100, centena; 1000, millar; 10000, dezena de millar; 100000, centena de millar; 1000000, cuento. The term *million*, however, had not yet been introduced, and it has not been fully ascertained at what time this introduction took place. Bishop Tonstall, 1522, in discussing the Latin nomenclature of numbers, speaks of the term *million* as in common use, but rejects it as barbarous, being used only by the vulgar; and Dr. Peacock remarks that by the vulgar he may have meant the arithmetical writers of England and other countries.

Stevinus divided numbers into periods of three places, called each period *membres*, and distinguished them as *le premier membre, le seconde membre*, etc. Instead of *million* he says *mille mille;* for a *thousand million* he uses *mille mille mille;* and for a *million million* he uses *mille mille mille mille.* It would appear from the practice of Stevinus, and from the observation of his contemporary, Clavius, that the term *million* was not at this time in general use amongst mathematicians. Albert Girard divides numbers into periods of six places, which he terms *première masse, seconde masse, troisième masse*, etc., the first of which only is divided into periods of three places each; but he does not use the word million. The term, however, was introduced into Recorde's arithmetic, and subsequently appeared in all succeeding English authors. It appears to have been admitted into German works much later than into the French and English. Kæstner says he found it in no German author on arithmetic in the first half of the 16th century; and Clavius is the first writer of that nation who has noticed the term, though he does not seem to have carried the innovation further, since he expresses *billions* by *milliones millionum*, which is the highest number he has occasion to use.

FUNDAMENTAL OPERATIONS.—The fundamental operations of arithmetic were, without doubt, invented by the Hindoos at a very early period. The work from which our knowledge of Hindoo arithmetic has been mainly derived, is the *Lilawati* of Bhascara, who lived about the middle of the 12th century. The work is named after the author's daughter, Lilawati, who, it appeared, was destined to pass her life unmarried and remain without children. The father, however, having ascertained a lucky hour for contracting her in marriage, left an hour-cup on a vessel of water, intending that when the cup should subside, the marriage should take place. It happened, however, that the girl, from a curiosity natural to children, looked into the cup to see the water coming in at the hole, when, by chance, a pearl separated from her bridal dress, fell into the cup, and rolling down to the hole, stopped the influx of water. When the operation of the cup had thus been delayed, the father was in consternation; and, examining, he found that a small pearl had stopped the flow of the water, and the long expected hour was passed. Thus disappointed, the father said to his unfortunate daughter, "I will write a book of your name, which shall remain to the latest times.—for a good name is a second life, and the groundwork of eternal existence."

This work frequently quotes Brahmegupta, an author who is known to have lived in the early part of the 7th century, and portions of whose works, containing treatises on arithmetic and mensuration, are still extant. Brahmegupta also refers to an earlier author, Arya-bhatta, who wrote an algebra and arithmetic, at least as early as the 5th century, and probably at a much earlier period, and who is considered the oldest of the uninspired and merely human writers among the Hindoos. It is thus clear, that Hindoo algebra and arithmetic are at least as ancient as Diophantus, and preceded, by four centuries, the introduction of these sciences among the Arabs; and it is equally clear, that the Arabs obtained their knowledge of the sciences from the Hindoos, and by means of their schools and universities aided in introducing them to the scholars of Europe. In tracing the history of the operations of arithmetic, we must therefore begin with the *Lilawati.*

The fundamental operations of arithmetic, as given in the *Lilawati*, are eight in number; namely, addition, subtraction, multiplication, division, square, square root, cube, cube root. To the first four of these the Arabs added two, namely, *duplation* and *mediation* or *halving*, considering them as operations distinct from multiplication and division, in consequence of the readiness with which they were performed; and they appear as such in many of the arithmetical books in the 16th century.

Addition.—The rule given in the *Lilawati* for addition is as follows: "The sum of the figures, according to their places, is to be taken in the direct or inverse order," which is interpreted to mean, "from the first on the right towards the left, or from the last on the left towards the right." In other words, they commenced indifferently with the figures in the highest or lowest places, a practice which would not lead to much inconvenience in their mode of working. Thus, to add 2, 5, 32, 193, 18, 10, 100, they proceed as follows:

Sum of the units, 2, 5, 2, 3, 8, 0, 0,	20
Sum of the tens, 3, 9, 1, 1, 0,	14
Sum of the hundreds, 1, 0, 0, 1,	2
Sum of the sums,	360

Subtraction.—The process of subtraction was also commenced either at the right or the left, but much more commonly at the latter; and it is remarkable that this method of beginning to subtract at the highest place, which is subject to considerable inconvenience, should have been so general. It is found in Arabic writers, in Maximus Planudes, a Byzantine writer of about the middle of the 13th century, and in many European writers as late as the end of the 16th century.

In Planudes, numbers to be added or subtracted are placed one underneath another, as in modern works on arithmetic; and the sum or difference is written above these numbers. When a term in the subtrahend is greater than the corresponding one in the minuend, a unit is written beneath them, as in the example in the margin.

18769	rem.
54612	min.
35843	sub.
1111	

In performing the operation, 3 is increased by the unit in the next place to the right, and also 5, 8, 4, and the terms thus increased are subtracted from the terms above, increased by 10, to find the remainder.

06779	rem.
2991	
30024	min.
23245	sub.

In other cases, the numbers are arranged, as in the margin, the digits 3, 0, 0, 2 in the minuend being replaced by 2, 9, 9, 1, and then 5 is subtracted from 4, 4 from 1, 2 from 9, 3 from 9, and 2 from 2, in order to get the remainder. It is obvious, that when such a preparation is made, it is indifferent where we commence the operation.

Bishop Tonstall attributes the invention of the modern practice of subtraction to an English arithmetician of the name of Garth. This method he has illustrated with great detail, and added, for the assistance of the learner, a *subtraction table*, giving the successive remainders of the nine digits when subtracted from the series of natural numbers from 11 to 19 inclusive, the only cases which can occur in practice.

In speaking of the methods of preceding writers, he has presented the example in the margin, in which it will be seen that the numbers from which the subtraction is actually made, are placed above the terms of the minuend.

2	9	10	10
3	0	1	0
1	1	1	1
1	8	9	9

In the arithmetic of Ramus, which was published in 1584, though written at an earlier period, we find the operation performed from left to right, and this method is followed by some other writers. Thus, in subtracting 345 from 432 the terms to be subtracted and the remainder are written as in the margin. When 3 is subtracted from 4, the remainder should be 1; but it is replaced by zero, since the next term in the subtrahend is greater than the corresponding term of the minuend; in the second term the remainder, which should be 9, is reduced to 8, since 5, the next term of the subtrahend, is greater than 2, the term above it, but the last remainder 7, is not changed.

	8	7
4	3	2
3	4	5

Orontius Fineus, the predecessor of Ramus in the professorship of Mathematics at Paris, in his *De Arithmetica Practica*, 1555, subtracts according to the method now used; and it is difficult to account for the adoption by Ramus of so inconvenient a method as he employed, when the method of Fineus must have been familiar to him, unless we attribute it to that love of singularity which led him to aspire to the honor of founding a school of his own.

Multiplication.—The author of *Lilawati* has noticed six different methods of multiplying numbers, and two others are mentioned by his commentators. These may be illustrated by their application to the following example: "Beautiful and dear Lilawati, whose eyes are like a fawn's, tell me what are the numbers resulting from one hundred and thirty-five taken into twelve? If thou be skilled in multiplication, by whole or by parts, whether by division or separation of digits, tell me, auspicious woman, what is the quotient of the product divided by the same multiplier?"

Here the multiplicand is 135, and the multiplier 12; and the first method, which consists of multiplying the terms of the multiplicand successively by the multiplier, is indicated in the margin.

1	3	5
12	12	12
	12	60
	3	6
	16	20

The second method, which consists in subdividing the multiplier into parts, as 8 and 4, and severally multiplying the multiplicand by them, is also indicated in the margin.

135	8	1080
135	4	540
		1620

The third method, which consists in separating the multiplier 12, into its two factors, 3 and 4, and multiplying successively by these factors, the last product being the result, is also represented in the margin.

135	4	20	540	3	120
		12			15
		4			1620
		540			

The fourth method consists in taking the digits as parts, viz., 1 and 2, the multiplicand being multiplied by them severally, and the products being added together according to the places of the figures, as is represented in the margin.

135	135
1	2
	270
	135
	1620

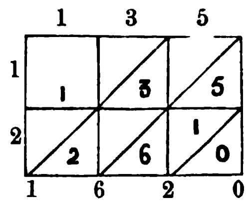
The fifth method consists in multiplying the multiplicand by the multiplier less 2, namely, 10, and adding the result to twice the multiplicand, as may be seen in the margin.

135	10	1350
135	2	270
		1620

The sixth method consists in multiplying the multiplicand by the multiplier increased by 8, namely, 20, and subtracting 8 times the multiplicand, as represented in the margin.

135	20	2700
135	8	1080
		1620

The other two methods are given in the Commentary of Ganesa. The first of these, which is represented in the margin, appears to have been very popular in the East, and was adopted by the Arabs, who termed it *shabacah*, or *net-work*, from the *reticulated* appearance of the figure which it formed, and also by the Persians under a slight alteration of form. It is found likewise in the works of the early Italian writers on algebra, and the same principle may be recognized in the process of multiplication by Napier's rods.



The second of these two methods of multiplication, as represented in the margin, is described in full by Ganesa. He, however, considers this method difficult, and not to be learned by dull scholars without oral instruction.

$$\begin{array}{r} 135 \\ \underline{12} \\ 10 \\ 11 \\ 5 \\ \underline{1} \\ 1620 \end{array}$$

The number and variety of these methods would seem to show that the operation of multiplication was regarded as difficult, and it is remarkable that the method now used is not found amongst them. We find no notice of the multiplication table among either them or the Arabs. At all events, it did not form a part of their elementary system of instruction, a circumstance which would account for some of the expedients which they appear to have made use of, for the purpose of relieving the memory from the labor of forming the products of the higher digits with each other.

The Arabs adopted most of the Hindoo methods of multiplication, and added some others of their own; among which are some peculiar contrivances for the multiplication of small numbers. They may also be considered as the authors of the method of *quarter squares*, or of finding the product of two numbers by subtracting the square of half their difference from the square of half their sum. The Arabs were most probably the inventors of the method of proof by casting out 9's, which is as yet unknown to the Hindoos; they called it *tarazu*, or the *balance*.

The work of Planudes was chiefly collected from the Arabic writers, as appears from his being acquainted with the method of casting out 9's. In multiplication he has chiefly followed the method of multiplying *crosswise* or κατὰ τὸν χιασμὸν, from the figure χ, which is employed to connect the digits to be multiplied together. Thus, in multiplying 24 into 35, we should write the factors as in the margin; and then multiply 4 into 5 (μοναδες), write down 0 and retain 2 for the next place; multiply 4 into 3, and 3 into 5, the sum is 22, which added to 2, makes 24 (δεκαδες), write down 4 and retain 2; lastly, multiply 2 into 3, add 2, which makes 8 (εκατονταδες), and the product is 840. He also gives another method which he acknowledges to be very difficult to perform with ink upon paper, but very commodious on a board strewed with sand, where the digits may be readily effaced and replaced by others. Thus, taking the same example, we multiply 2 into 3, write 6 above the 3; multiply 2 into 5, the result is 10; add 1 to 6, and replace it by 7, or write 7 above it; multiply 4 into 3, the product is 12; write 2 above 5, and add 1 to 7, which is replaced by 8, or 8 written above it; lastly, multiply 4 into 5, the result is 20; add 2 to 2, place 4 above it and after it the cipher; the last figures, or those which remain without accents, will express the product required.

840
35
χ
24

840
7′
6′2′
3 5
2 4

Division.—The extreme brevity with which the rules of division are stated in the *Lilawati* renders it difficult to describe the Hindoo method of dividing numbers. We are directed *to abridge the dividend and divisor by an equal number, whenever that is practicable;* that is, to divide them both by any common measure; thus, instead of dividing 1620 by 12, we may divide 540 by 4, or 405 by 3. We find, however, in one of the commentators on this work, a description of the process of long division, which, if exhibited in a scheme, would exactly agree with the modern rule

ITALIAN METHODS.—The Italians, who cultivated arithmetic

4

with so much zeal and success, from a very early period adopted from their Oriental masters many of their processes for the multiplication and division of numbers; adding, however, many of their own, and particularly those which are practiced at the present time. In the *Summa de Arithmetica* of Lucas di Borgo, we find eight different methods of multiplication, some of which are designated by quaint and fanciful names. We shall mention them in their order.

1. *Multiplicatio: bericuocoli e schacherii.* The second of these names is derived from the resemblance of the written process to the squares of a chess-board; the first from its resemblance to the *checkers* on a species of sweetmeat or cake, made chiefly from the paste of *bacochi* or apricots, which were commonly used at festivals. The process is exhibited in the margin. This method is presented by Tartaglia and later Italian writers without the squares; and it thus became the method which is now universally used, and which was adopted from the beginning of the 16th century by all writers on arithmetic, nearly to the exclusion of every other method.

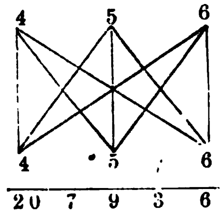
			4	5	6
			3	7	8
		3	6	4	8
	3	1	9	2	
1	3	6	8		
1	7	2	3	6	8

2. *Castelluccio; by the little castle.* This method, as indicated in the margin, uses the upper number as the multiplier, and begins with the higher terms. This method was much practiced by the Florentines, by whom it was sometimes called *all' indietro,* from the operation beginning with the highest places, *more Arabum,* according to the statement of Pacioli.

9876
6789
61101000
5431200
475230
40734
67048164

3. *Columna, o per tavoletta; by the column, or by the tablets.* These were tables of multiplication, arranged in columns, the first containing the squares of the digits, the second the products of 2 into all digits above 2; the third, of 3 into all digits above 3; and so on, extending in some cases as far as the pro-

ducts of all numbers less than 100 into each other. Pacioli says that these tablets were learned by the Florentines, and their familiarity with them was considered by him as a principal cause of their superior dexterity in arithmetical operations. This method is used in multiplying any number, however large, into another which is within the limits of the table. Thus, to multiply 4685 by 13, the terms of the multiplicand are multiplied successively by 13, and the results formed in the ordinary manner.

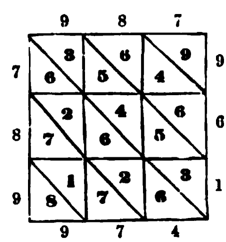


4. *Crocetta sive casella; by cross multiplication.* This method is said to require more mental exertion than any other, particularly when many figures are to be combined together. Pacioli expresses his admiration of this method, and then takes the opportunity of enlarging on the great difficulty of attaining excellence, whether in morals or in science, without labor.

5. *Quadrilatero; by the square.* This is a method which has been characterized as elegant, and as not requiring the operator to attend to the places of the figures when performing the multiplications. The method is represented in the margin, and will be readily understood.

	5	4	3	
	5	4	3	
1	6	2	9	
2	1	7	2	9
2	7	1	5	4
	2	9	4	8

6. *Gelosia sive graticola; latticed multiplication.* "It is so called," says Pacioli, "because the disposition of the operation resembles the form of a *lattice*, a term by which we designate the blinds or gratings which are placed in the windows of houses inhabited by ladies so that they may not easily be seen, as well as by other *nuns*, in which the lofty city of Venice greatly abounds." The method will be readily understood by the example given in the margin, which multiplies 987 by 987. It is the same as one previously



noticed, which was in common use among the Hindoos, Arabians, and Persians.

7. *Ripiego;* multiplication by the *unfolding* or resolution of the multiplier into its component factors. Thus, to multiply 157 by 42, resolve 42 into its *ripieghi* or *factors*, 6 and 7, and multiply successively by them.

8. *Scapezzo;* multiplication by *cutting up*, or separating the multiplier into a number of parts, which compose it by addition. Thus, to multiply 2093 by 17, we separate 17 into 10 and 7, multiply by each, and take the sum of the products. In some cases both multiplicand and multiplier were separated into parts. Thus, the multiplication of 15 by 12 was performed as in the margin.

4,	5,	6	
2,	4,	6	
8	16	24	30
10	20	30	60
12	24	36	90
30	60	90	180

In another Italian arithmetic, published in 1567, by Pietro Cataneo Sienese, we find the same distinctions preserved, and the same names, or nearly so, attached to them; the method of cross multiplication is expressly attributed to Leonard of Pisa, who derived it, in common with Maximus Planudes, from the Hindoos, through the Arabians. It is not impossible that St. Andrew's cross, which is the sign of multiplication, was derived from the custom of uniting the numbers to be multiplied together by lines which crossed each other, as in the example given in the margin.

5 9
×
4 7
2773

Both Lucas di Borgo and Tartaglia mention other methods of multiplication which were made use of in their time. An extraordinary passion seems to have prevailed in that age for the invention of new forms of multiplication, and every professional practitioner of arithmetic considered it as an important triumph of his art if he could produce a figure more elegant and more refined in its composition and arrangement than those which were used by others. They are, all of them, however, characterized by Pacioli as inconvenient, at least compared with those which he had given; and Tartaglia treats them as trifling and superfluous, such as any one may invent who is acquainted with the 2d proposition in the 2d Book of Euclid.

The Hindoos, as has been stated, had no proper knowledge of the multiplication table, and the Arabs do not appear to have made use of the table of Pythagoras as the basis of their arithmetical education; the credit of introducing it, therefore, is due to the early Italian writers on the science, who probably found it in the writings of Boethius, and adopted it thence. Even after the Italian arithmeticians were familiar with this table, many writers of other countries considered it important to relieve the memory from the labor of retaining it for the products of all digits exceeding 5, by giving rules for their formation. The principal rule for this purpose, called *regula ignavi*, or the *sluggard's rule*, was adapted from the Arabians, and is found in Orontius Fineus, Recorde, Laurenberg, and most other writers between the middle of the 16th and 17th centuries. The rule is as follows: *Subtract each digit from 10, and write down the difference; multiply these differences together, and add as many tens to their product as the first digit exceeds the second difference, or the second digit the first difference.* The Arabians made use of this and other similar rules which applied to numbers of two places of figures, a practice which may be accounted for by their very general use of sexagesimals, and the consequent importance of being able to form the products which are found in a sexagesimal table.

7 3	8 2	9 1
×	×	×
6 4	7 3	8 2
42	56	72

Many other expedients were proposed to relieve the memory, in the process of multiplication, from the labor of carrying the tens. An interesting one is presented by Laurenberg, an author who endeavored to elevate the character of the common study of arithmetic by collecting all his examples from classical authors, and by making them illustrative of the geography, chronology, weights and measures of antiquity. It will be understood from the example given, without explanation.

```
  5142
    43
  ----
   106
  1532
   108
 2046
 ------
 221106
```

Division.—Neither Planudes nor the early Arabic writers seem to have presented any methods of dividing that merit the special notice of the writers on the history of arithmetic. Lucas di Borgo gives four distinct methods which we proceed to explain. These methods had particular names, as in multiplication.

1. *Partire a regolo*, sometimes called also *partire per testa* or *division by the head*, was used when the divisor was a single digit, or a number of two places, such as 12, 13, etc., included in the *librettine* or Italian tables of multiplication. The method will be readily understood from the example given. Di Borgo says: "This method of division is called by the vulgar, the *rule*, from the similitude of the figure to the carpenter's rule which is made use of in the making of dining-tables, boxes, and other articles, which rules are long and narrow."

6
3478
$579\frac{4}{6}$

2. *Per ripiego;* which consists in resolving the divisor into its simple factors, or *ripieghi*. It will be readily understood from the example given, and be recognized as a common method of modern arithmetics.

	63
7	250047
9	35721
	3969

3. *A danda;* which the author says is thus called for reasons which will be readily seen in the operation itself, which represents the division of 230265 by 357, giving a quotient of 645. The process is the same as our common method of long division, only the numbers are not so conveniently written. It was called *a danda*, or *by giving*, because after every subtraction we give or add one or more figures on the right hand. The author, however, prefers the next method.

Divisor.	Proveniens.
357	645
230	265
2142	
1606	
1428	
1785	
1785	

4. *Galea vel galera vel batello ;* so called from the process resembling a *galley*, "the vessel of all others most feared on

the sea by those who have good knowledge of it; the most secure and swiftest; the most rapid and lightest of the boats that pass on the water." The method may be illustrated by dividing 97535399 by 9876. We first write the dividend, and underneath it the divisor, and commence with the second figure of the dividend, since the divisor is not contained in the first four terms of the dividend.

```
  86
 975
16301
97535399(9
 9876
```

Multiplying the divisor by the first term of the quotient, 9 times 9 are 81, which subtracted from 97 leaves 16, which is written above 97; then cancel 97 and 9 in the divisor; 9 times 8 are 72, which taken from 165, leaves 93; write 9 above 16 and 3 above 5 in the dividend, and cancel 165, and 8 in divisor; 9 times 7 are 63, which subtracted from 933 leaves 870; cancel 933 in remainder, and 7 in divisor; 9 times 6 are 54, which subtracted from 705 leaves 651; cancelling 705, and 6 in the divisor, we have as a remainder 8651399. For multiplying by the second quotient figure, we arrange the divisor as in the margin, and proceed as before. The complete operation is represented by the last work in the margin, and is so apparent that it needs no further explanation.

```
  86
 975
16301
97535399(98
 98766
  987
```

```
   15
   765
   829
  14544
  861022
  975565
 16301573
 97535399(9876
  9876666
   98777
    988
     9
```

Tartaglia states that it was the custom in Venice for masters to propose to their pupils as the last proof of their proficiency in this process of division, examples which would produce the complete form of the galley, with its masts and pendant. The last addition to the work was supplied by the scheme for the proof of the accuracy of the operation by casting out the 9's. Dr. Peacock gives an example showing the numbers thus arranged, which is very curious, but too long for insertion here.

The same process is illustrated by an example from the numerous calculations by Regiomontanus, in his tract on the quadrature of the circle, written as early as 1464, though not published until 1532. The question proposed is to divide 18190735 by 415. The divisor is placed under the dividend and repeated at every step backward, and all the figures erased in succession. The quotient, 43833 is placed down the side and along the bottom, the remainder 40 being the only digits left on the board.

11	
3134	
154750	4
276548	3
18190735	8
4155555	3
41111	3
444	
43833	

It is amusing to observe the enthusiastic admiration of Di Borgo for this method of division. When describing the preceding method he seems impatient, and looks forward with pleasure to the description of the method *a la galea*, as possessing a certain charm and solace, remarking that it is a noble thing to see in any species and scheme of numbers, a galley perfectly exhibited, so as to be able to observe its mast, its sail, its yards and its oars, launched in the spacious ocean of arithmetic. This method, we are surprised to learn, appears to have been preferred by nearly every writer on arithmetic as late as the end of the 17th century. It was adopted by the Spaniards, French, Germans, and English; and it is the only method which they have thought necessary to notice. It is found almost universally in the works of Tonstall, Recorde, Stifelius, Ramus, Stevinus, and Wallis; and it was only at the beginning of the 18th century that this method of division, called by the English arithmeticians the *scratch* method of division, from the *scratches* used in cancelling the figures, was superseded by the method now in common use, which was specifically called Italian division, from the country whence it was derived.

Recorde noticed the Italian method of division, which, he says, "I first learned of, and is practiced by my ancient and especial loving friend, Master Henry Bridges, wherein not any one figure is cancelled or defaced. He illustrates the method by an example which we subjoin; though, as before stated, he preferred the *scratch* method of dividing.

```
33)7890(239 3/33
   66
   129
    99
   ---
   300
   297
   ---
     3
```

POWERS AND ROOTS.—The author of the *Lilawati* has given rules for the formation of squares and cubes, as well as for the extraction of the corresponding roots. The rule for the formation of the square, which is very ingenious, is as follows: Place the square of the last digit over the number, and the rest of the digits doubled and multiplied by the last are to be placed above them respectively; then repeating the number with the omission of the last digit, perform the same operation. This is illustrated in squaring the number 297.

```
4
36
 81
 28
 126
   49
 -----
   297
 -----
 88209
```

In performing the converse operation, every uneven place is marked by a vertical line, and the intermediate digits by a horizontal one; but if the place be even, it is joined with the contiguous odd digit. It may be illustrated by extracting the square root of 88209, enough of the work being indicated to show the nature of the method. We subtract from the last uneven place, 8, the square 4, and there remains 48209, represented as in the margin. Double the root 2, making 4, and divide 48, the number denoted by the next two terms, by the result, obtaining 9 (10 would be too large), and subtracting 9 times 4 or 36, we have 12209. From the uneven place, with the residue, 122, subtract the square of 9, or 81; the remainder is 4109 Double 9, giving 18, and unite the result with 4, giving 58, and divide 410 by it, and we have 7, and the remainder,

```
—|—|
8 8 2 0 9
—|—|
4 8 2 0 9
    —|
1 2 2 0 9
    —|
  4 1 0 9
    etc.
```

49, to which the square of the quotient 7, or 49, answers without a residue. The double of the quotient, 14, is put in a line with the preceding double number, 58, making 594, the half of which is the root sought, 297.

This account of the Hindoo method of extracting square root, is taken from the commentators on the *Lilawati*, and does not differ essentially from the method now used; and the same may be said of the method of extracting the cube root, the principal difference from the present method being found in their peculiar methods of multiplying and dividing.

The method of extracting the square root used by the Arabians resembled their method of division; and it is probable that they are both founded on the Greek methods of performing these operations with sexagesimals. The example given will show the form of operation. Vertical lines being drawn and the numbers distinguished into periods of two figures, the nearest root of 10 is 3, which is placed both below and above, and its square, 9, subtracted; the 3 is now doubled, and 6 being written in the next column, is contained twice in 17, or the remainder with the first figure of the next period; the 2 is therefore set down both above and below, and being multiplied into 6 gives 12, which is subtracted from 17, leaving 5; the square of 2, or 4, is now subtracted from 55, and 518, the remainder, with the succeeding figure, is divided by 64, or the double of 32, giving 8 for the quotient; then 8 times 64 are 512, which, subtracted from 618 leaves 6; and 64 is exhausted by taking from it the square of 8. It is said that this mode was adopted from the Arabs by the Hindoos

	3		2		8
1	0	7	5	8	4
	9				
	1	7			
	1	2			
		5	5		
			4		
		5	1	8	
		5	1	2	
				6	4
				6	4
				4	8
		6	6		
	3		2		

The earlier mathematicians of Europe employed a similar method of extracting the square root, though perhaps not quite so systematic and regular. In proof of the rule which they followed, they constantly refer to the 4th proposition of the 2d book of Euclid. I will give several examples illustrating their methods.

The first is from the arithmetic of Pelletier, the first edition of which was published in 1550. It represents his method of extracting the square root of 92416, and is so simple it needs no explanation. It will be seen that the dots marking the periods into which the number is separated are placed under the number, instead of above it as is now the custom.

```
 0
 92416
  604
    4(304
 ----
 2416
```

The second example is from the work of Lucas di Borgo, and is in the form of the process which was most commonly adopted. The example, as will be seen, is the extraction of the square root of 99980001. The scheme will require no explanation, but will be readily understood by those who are familiar with the *galley* form of division.

```
   00
   018
   1270
  20880
 09969800
 18778980
 99980001(9999
  9898989
  11999
    1
```

We present another illustration taken from the tract, already mentioned, of Regiomontanus. The question is to find the square root of the number 5261216896. Now the nearest square to 52 is 49, leaving 3 to be set above the 2, while 7, the root, is placed in the vertical line; then double of 7, or 14, being set under the 36, is contained twice, and 2 is accordingly placed under the 7; but twice 1 is 2, which taken from 3 leaves 1, and twice 4 are 8, which taken from 6, or 16, leaves 8, and extinguishes the 1 before it; and twice 2 are 4, which taken from 1, or 11, leaves 7, and converts the preceding 8 into 7. In this way the process advances till the

```
    123
   2465
  1757174   |7
  38796595  |2
 5261216896 |5
  14406     |3
   430      |4
   145      |
    14      |
     1      |
 ---------- |
 72534      |
```

figures become successively effaced. The root, 72534, is placed both at the right hand side and also immediately below the work. The divisors do not appear to be right, but we do not feel sufficiently acquainted with the subject to change them, and do not possess the original work by which we can verify them.

The method of extracting cube root used by the Arabians and Persians, and by them communicated to the Hindoos, resembles likewise their method of performing division. We will illustrate it by extracting the cube root of 91125. Having drawn the vertical lines as indicated, the several digits of the number are inscribed between them, and dots set over the first, fourth, seventh, etc., reckoning from the right. The nearest cube to 91 is 64, which is set down and subtracted, leaving 27. To obtain the next term of the root, 3 times 16, which is 3 times the square of the root found, is written below, and being contained 5 times in 271, the divisor is completed by adding 3 times the product of 4 and 5, or 60, and then the square of 5, or 25, making in all 5425, each term of which is multiplied by 5, and the products subtracted in succession.

	4			5
9	1	1	2	5
6	4			
2	7			
2	5			
	2			
	2	0		
		1		
		1	0	
			2	
			2	5
	4	8		
		6	0	
			2	5
	5	4	2	5

The ancient mode of extracting the cube root practiced in Europe was similar to the process just explained, but not so regular and formal. The annexed example is taken from the *Ars Supputandi* of the famous Cuthbert Tonstall, Bishop of Durham, the earliest treatise on arithmetic published in England, and a work of no common merit. The number 250523582464,

4′ 7′6′
3′ 4′0′
2′5′0′5′2′3′5′8′2′4′6′4′
6 3 0 4
3′4′1′8′7′8′9′8′9′0′4′
1′0′2′1′5′9′
1′2′2′5′1′ 0′
4′9′6′ 8′2′4′6′
7′

whose root is to be extracted, is placed above two parallel lines, between which the root 6304 is inserted; the successive divisors and the corresponding remainders being written alternately below and above, and the figures erased as fast as the operation advances, the operation of erasure being here denoted by accents.

Stifelius, who usually sought to generalize the methods of his predecessors, has considered the process of extracting the square root in connection with those of higher powers. By observing the formation of the powers themselves, he discovered certain schemes, or *pictures* as he calls them, for extracting the square, cube, biquadrate, etc., roots. If we indicate the terms of a binomial root by a and b, his scheme for the square root would consist of a-20-b and b^2 written under the b to denote addition. The meaning of the scheme is that in extracting the square root, the first term, a, must be multiplied by 20 to get the divisor from which we determine the second term, b; after which the sum of the product of a, 20, and b, and b^2 must be subtracted from the first remainder. His method is illustrated by the extraction of the square root of 6765201, as here given.

```
2
6765201(2601
2 - 20 - 6.
         36-276
26 - 20   0-0
2-60-20   1
          1-5201
```

The history of the origin of these arithmetical processes is derived from Prof. Leslie and Dr. Peacock, much of it having been copied word for word from the originals. The origin of methods in *Fractions, Decimals, Rule of Three, Continued Fractions*, etc., will be given in connection with those subjects; and such other historical information as it is thought will be of interest to the reader will be presented in its appropriate place. Occasionally the same fact is repeated, in order to give a completeness to the particular subject discussed.

Part I.

THE NATURE OF ARITHMETIC

SECTION I.

THE NATURE OF NUMBER.

SECTION II.

ARITHMETICAL LANGUAGE.

SECTION III.

ARITHMETICAL REASONING

SECTION 1.

THE NATURE OF NUMBER.

I. Subject Matter of Arithmetic

II. Definition of Number

III. Classes of Numbers.

IV. Numerical Ideas of the Ancients

CHAPTER I.

NUMBER, THE SUBJECT MATTER OF ARITHMETIC.

NUMBER was primarily a thought in the mind of Deity. He put forth His creative hand, and number became a fact of the universe. It was projected everywhere, in all things, and through all things. The flower numbered its petals, the crystal counted its faces, the insect its eyes, the evening its stars, and the moon, time's golden horologe, marked the months and the seasons.

Man was created to apprehend the numerical idea. Finding it embodied in the material world, he exclaimed, with the enthusiasm of Pythagoras, "Number is the essence of the universe, the archetype of creation." He meditated upon it with enthusiasm, followed its combinations, traced its relations, unfolded its mystic laws, and created with it a science—the beautiful science of Arithmetic. Let us consider the origin and nature of the idea out of which man has created this science of exact relations and interesting principles.

Origin.—The conception of number begins with the contemplation of material objects. Objects are found in combinations or collections, and the inquiry, *how many* of such a collection, gives rise to the idea of number. The young mind looks out upon nature, communes with its material forms, sees unity and plurality, the one and the many, all around it, and awakens to the numerical idea. Strange law of spiritual development! the material thing calls into being the immaterial thought. The unity and plurality, as it dwelt in the God-mind and was embodied in the material world, passes over to the mind of man, and appears as an idea of the immaterial spirit.

The idea of definite numbers is developed by a mental act called *counting*. We ascertain the how-many of a collection, by counting the objects in the collection. The act of counting, (*one*, *two*, *three*, etc.), is the foundation of all our knowledge of number. In counting, we pass in succession from one object to another. Succession implies time, and is only possible in time. The idea of number, therefore, has its origin in the fact of time, and is possible only in this great fact. A brief consideration of this relation will not be uninteresting.

Time is one of the two great infinitudes of nature. Space and Time are the conditions of all existence. Time enables us to ask the question, *when;* Space, the question, *where.* Space is the condition of matter regarded as extended, and is thus the condition of extension. Extension has three dimensions, length, breadth, and thickness. The science of extension is geometry. Space is thus seen to be the basis or condition of the science of geometry.

Time is the condition of events, as Space is of objects. Every event exists in Time, as every object must exist in Space. Time has somewhat the same relation to the world of mind, that Space has to the world of matter. Matter extends in Space, as mind protends in Time. This intimate relation of Number and Time leads me to present a few thoughts concerning the nature of Time, and the development of the idea of Number from it.

Time is not a mere *abstraction*. It is not a quality perceived in an object and drawn away from it by the power of abstract thought, and conceived as an abstract notion. Neither is it a *general idea*, or a *concept*. We do not first get particular notions of Time, and then, by putting these together, form a general idea of it. No summation of particular *times* can give the grand, unlimited idea of Time that the mind possesses. Indeed, we do not consider particular *times* as examples of

Time in general; but we conceive all particular *times* to be parts of a single endless Time. This continually flowing and endless time is what offers itself to us when we contemplate any series of occurrences. All actual and possible *times* exist as parts of this original and general Time. Therefore, since all particular *times* are considered as derivable from time in general, it is manifest that the notion of time in general, cannot be derived from the notions of particular times.

Time is a grand intuition. It is an idea which is formed in the mind when the proper occasion of sensible experience is presented. Sensible experience is not the cause, but the occasion upon which the mind conceives or originates this idea. It is the product of the higher intuitive power known as the Reason. But Time is not only an idea, it is a great reality. It has a real objective existence, independent of the mind which conceives it. Were there no minds to conceive it, time would still exist as the condition of events. Were all events blotted out of existence, time would remain an endless on-going.

Time is infinite. No mind can conceive its beginning; no mind can conceive its end. All limited times merely divide, but do not terminate the extent of absolute time. In it every event begins and ends, while it never begins and never ends. It is, in its very nature, like Him who inhabiteth eternity, without beginning and without end.

Time gives rise to succession, as space does to extension. Out of succession grows the idea of Number, and the science of Number is Arithmetic. Arithmetic, therefore, has somewhat the same relation to time, that geometry has to space. In view of this fact, some philosophers have called geometry the science of space, and arithmetic the science of time. This view of arithmetic, however, has not been adopted by all writers, since there are other ideas growing out of time than that of number. Whewell, in writing of the Pure Sciences, speaks of the three great ideas—*Space, Time*, and *Number;* thus distinguishing between Number and Time. Several efforts have been made to construct a science of Time; the most remarkable is that of Sir William Rowan Hamilton, which resulted in the invention of the wonderful Calculus of Quaternions.

Time is considered as having but one dimension. In this respect it differs from Space, which has three dimensions, length, breadth, and thickness. Time may be regarded as analogous to a line, but it has no analogy to a surface or a volume. Time exists as a series of instants which are before and after one another; and they have no other relation than this of *before* and *after*. This analogy between Time and a line is so close, that the same terms are applied to both ideas, and it is difficult to say to which they originally belonged. Time and lines are called *long* and *short;* we speak of the *beginning* and the *end* of a line, of a *point* of time, and of the *limits* of a portion of duration.

There being nothing in Time which corresponds to more than one dimension of extension, there is nothing which bears any analogy with figure. Time resembles a line extending indefinitely both ways; all partial times are portions of this line; and no mode of conceiving time suggests to us a line making an angle with the original line, or any other combination which might give rise to figures of any kind. The analogy between time and space, which in many circumstances is so clear, here disappears altogether. Spaces of two and of three dimensions, surfaces and volumes, have nothing to which we can compare them in the conceptions arising out of time.

The conception which peculiarly belongs to time, as figure does to space, is that of the recurrence of times similarly marked. This may be called *rhythm*, using the word in a general sense. The forms of such recurrence are noticed in the versification of poetry and the melodies of music. All kinds of versification, and the still more varied forms of recurrence of notes of different lengths, which are heard in all the varied strains of melodies, are only examples of such modifications or configurations, as we may call them, of time. They involve relations of various portions of time, as figures involve relations of various portions of space. But yet the analogy between rhythm and figure is by no means very close; for in rhythm we have relations of quantity alone in parts of time, whereas in figure we have relations not only of quantity, but of a kind altogether different—namely, of position. On the other hand, a repetition of similar elements, which does not necessarily occur in figures, is quite essential in order to impress upon us that measured progress of time of which we here speak. And thus the ideas of time and space have each their peculiar and exclusive relations; position and figure belonging only to space, while repetition and rhythm are appropriate only to time.

One of the simplest forms of recurrence is *alternation*, as we have alternate accented and unaccented syllables. For example:

"Come one′, come all′, this rock′ shall fly′."

Or without any subordination, as when we reckon numbers, and call them in succession, *odd*, *even*, *odd*, *even*, etc.

But the simplest of all forms of recurrence is that which has no variety, in which a series of units, each considered as exactly similar to the rest, succeed one another; as *one*, *one*, *one*, and so on. In this case, however, we are led to consider each unit with reference to all that have preceded; and thus the series *one*, *one*, *one*, and so forth, becomes *one*, *two*, *three*, *four*, *five*, and so on; a series with which all are familiar, and which may be continued without limit. We thus collect from that repetition of which time admits, the conception of *Number*.

This conception of the origin of Number out of the idea of Time is now accepted by the philosophical world as correct. The subject is so happily treated by Whewell that I have taken the liberty of adopting some of his language in the above statement.

CHAPTER II.

DEFINITION OF NUMBER.

THE idea of number is so elementary that it is difficult to define it scientifically. Various definitions have been presented by different writers upon the subject, though no one has hitherto given one which is, in all respects, satisfactory. The two most celebrated definitions are those of Newton and Euclid, both of which will be briefly considered.

Newton defined number as "the abstract ratio of one quantity to another quantity of the same species." This definition is philosophical and accurate. It shows number to be a pure abstraction derived from a comparison of things. In discrete quantity, it regards one of the individual things as the unit of comparison; while in continuous quantity the unit is assumed to be some definite portion of the quantity considered.

This definition was no doubt primarily intended to apply to extended quantity, in which there is no natural unit, but in which some definite portion of the quantity is assumed as a unit of measure, and the quantity estimated by comparing it with this unit as a standard. Such comparison gives rise to three kinds of numbers; integral, fractional, and surd numbers. When the quantity measured contains the unit a definite number of times, the number is integral; when it is only a definite part of the measure, the number is fractional; when there is no common measure between the unit and the quantity measured, the number is a surd or radical.

The definition of Newton, though admirable in many respects, is not suitable for popular use. It is too abstract and difficult to be understood by young pupils; and cannot, therefore, be recommended for our elementary text-books. It may be said, also, that it does not express clearly the process of thought by which we attain the idea of number. It is more appropriate as applied to continuous than to discrete quantity, while the idea of number begins with discrete rather than continuous quantity. In this latter respect it may possibly be improved by changing the form of expression, while retaining its spirit: thus, *A number is the relation of a collection to the single thing.* This is simpler than the original form, and is in many respects a very satisfactory definition.

Euclid defined number to be "an assemblage or collection of units or things of the same species." This definition, slightly modified, has been generally adopted by mathematicians. In its original form it excluded the number *one*, since one thing is not an assemblage or collection, and hence it has been changed to read—*A number is a unit or a collection of units.* This is the definition which is now found in a large number of text-books.

This definition, however, is not strictly correct. A *number* is not precisely the same as a *collection* of units, and a collection of units is not necessarily a number. In other words, there is a difference between a *collection* of things and a *number* of things. This may be more clearly seen by the use of the corresponding verbs. *To collect* and *to number* are two different things. We may *collect* without numbering, and we may *number* without collecting; I may *collect* a *number* of things, and I may *number* a *collection* of things. If a basket of apples were strewn over the floor and I were told to collect them, I might do so without numbering them; or, if told to number them, I might do so without collecting them. In the latter case I would have a number of apples without having a collection of apples, except the mental collection, from which it appears that a number is not precisely the same as a collection. Number is more definite than collection. A collection is an indefinite thing, numerically considered; number is that which makes it definite. Number and collection are not, therefore, identical. Number is rather the *how many* of the collection. It is thus seen that Euclid's definition, as modified and now introduced into most of our text-books, is not without scientific objections. It must be admitted, however, that there is no other one word which so nearly expresses the idea of the word *number* as *collection;* and, for ordinary purposes, they may be used interchangeably. Thus we may say, in analysis, we pass from the *collection* to the *single thing;* from a *number* to *one.* It is, therefore, regarded as the best definition for the ordinary text-book, that has hitherto been presented.

From this discussion it will appear, as above stated, that it is difficult to present a good definition of Number. This difficulty is due to the fact that Number is a simple term expressing a simple idea, for which we have no other word of precisely the same signification. Simple terms are always difficult to define, from the very fact that they define themselves. Indeed, perhaps there is nothing in the way of a definition of number clearer than the identity—"A Number is a Number." The following, though liable to a verbal objection, seems to me to come as near the truth as anything that has yet been presented: A Number is *the how-many* of a collection of units; or, A Number is *how many* times a single thing is reckoned, or is contained in a collection.

The first excludes the number *one*, unless, as some writers propose, we give a special signification to collection. The second provides for the number *one*, but is not, in other respects, so satisfactory as the first. These definitions express precisely the idea of a number, but the use of the expression *how many* as a noun, is not elegant in the English language. The simplest and most satisfactory definition for a text-book is, "A Number is a unit or a collection of units."

The definitions of a number, as given in some of our text-books, are very objectionable. One author says: "Numbers are repetitions of units." This may answer as a popular statement, but is very far from meeting the requirements of a scientific definition. Another author says: "A number is a definite expression of quantity." So is a triangle or a circle, each of which should be a number if this definition is correct. Another says: "A number is an *expression* that *tells* how many." The two errors are, first, that a *number* is not an *expression;* and, second, that a number does not *tell* anything. The following definitions have also been given by different writers: "Number is a term signifying one or more units;" "A number is an expression of one or more things of a kind;" "A number is an expression of quantity by a unit, or by its repetition, or by its parts;" "Number consists of a repetition of units;" "A number is either a unit or composed of an assemblage of units;" "A number is a term expressing a particular sameness of repetition." Other definitions, equally incorrect, may be found by leafing over text-books upon the subject. A very simple definition, and especially suitable for a primary text-book is, "A number is one or more units." It may be remarked that authors seem to be adopting the definition of Euclid, with the modification presented above, so that the standard definition in our text-books is becoming, "A number is a unit or a collection of units."

To give a perfect definition of Number is exceedingly difficult, if not impossible. Stevinus defines it as "that by which the quantity of anything is expressed," but mathematicians have not adopted it. Euler's definition, "number is nothing else than the ratio of one quantity to another quantity taken as a unit," has been highly commended. "Number is a definite expression of quantity," has its advocates. "Number is quantity conceived as made up of parts, and answers to the question, How many?" has the authority of a very careful writer. The world, however, still waits for a simple and accurate definition, which may be generally adopted.

CHAPTER III.

CLASSES OF NUMBERS.

NUMBERS have been variously classified with respect to different properties, or by regarding them from different points of view. The fundamental classes to which attention is here called, are *Integers*, *Fractions*, and *Denominate Numbers.* These three classes are practically and philosophically distinguished, and constitute the basis of three principal divisions of the science of arithmetic. Logically, the distinction is not without exception, for a Fraction may be denominate, and a Denominate Number may be integral; but the division is regarded as philosophical, since they are not only different in character, but require distinct methods of treatment, and give rise to distinct rules and processes. The philosophical character and relation of these three classes of numbers, will appear from the following considerations:

Integers.—The Unit is the basis or beginning of numbers. A number is a synthesis of units; it is the *how-many* of a collection of units. These units, as they exist in nature, are whole things, undivided; hence the first numbers of which a knowledge is acquired, are whole numbers, that is, collections of entire or undivided units. Such units, being entire, are called integral units, and the numbers composed of them are called integral numbers, or *Integers.* An *Integer* is, therefore, a collection of integral units, or, as popularly defined, it is a whole number. It is a product of pure synthesis.

Fractions.—The Unit, as the basis of arithmetic, may be multiplied or divided. A synthesis of units, as we have seen, gives rise to Integers; a division of the unit gives rise to Fractions. Dividing the unit into a number of equal parts, we see that these parts bear a definite relation to the unit divided, and by taking one or more of these parts, we have a *Fraction.* It is thus seen that the conception of a fraction implies three things: first, a *division* of the unit; second, a *comparison* of the part to the unit; and third, a *collection* of the fractional parts. In other words it is the product of three operations, *division, comparison,* and *collection;* or, like the logical nature of the science of arithmetic itself, a fraction is a triune product, consisting of *analysis, comparison,* and *synthesis.*

Denominate Numbers.—The unit of a simple integral number exists in nature. A *Denominate Number* is a collection of units not found in nature; it is a collection of artificial units adopted to measure quantity of magnitude. The philosophical character of a denominate number is indicated in the following statement: Nature, regarded as *how many* and *how much,* gives rise to two distinct forms of quantity; quantity of *multitude,* and quantity of *magnitude.* Quantity of multitude is primarily expressed by numbers, since it exists in the form of individuals, or units; quantity of magnitude does not admit, primarily, of being expressed in numerical form. To estimate quantity of magnitude, we must fix upon some definite part of the quantity considered as a unit of measure, by which we can give it a numerical form of expression.

A *Denominate Number* may, therefore, be defined as *a numerical expression of quantity of magnitude.* Or, since the unit is a measure by which the quantity is estimated, we may define it to be *a number whose unit is a measure.* Again, since the unit is not natural but artificial, we may define it to be *a number whose unit is artificial.* Either of these definitions suffices to distinguish it from the other two classes of numbers. It differs from them in respect of the nature of the quantity to which it refers, and also in its origin and composition. In the simple integral numbers, the units, as found in nature, are collected; in the denominate number, the unit is assumed, the quantity compared with the unit, and the result expressed numerically. The same kind of quantity may be measured by different units, bearing a definite relation to each other, which gives rise to a *scale of units*. Taking our scales as they now exist, we have a series of units definitely related to each other, forming a *Compound Number*, which does not appear in the other classes of numbers. This, however, is rather incidental than essential, as it partially vanishes when we apply the decimal scale to quantity of magnitude, as in the metric system of weights and measures.

It is thus seen that there are three distinct classes of numbers; and, since they require different methods of treatment, they will be considered independently. The remainder of this chapter will be devoted to the discussion of some of the peculiarities of integral numbers.

Classes of Integers. — Simple Integral Numbers, being learned before Fractions and Denominate Numbers, are the first class to which the term *number* was applied; they have consequently appropriated to themselves the almost exclusive use of the word *number*. Thus, it is the general custom to speak of Numbers, Fractions, and Denominate Numbers, apparently forgetful that they are all numbers. This custom being so common, the word *Integer* being somewhat inconvenient, and some of the properties which belong to integral numbers applying also to the other two classes, I will also use the word *number* in place of *integral number* in considering this part of the subject.

Numbers are of two general classes, *Concrete* and *Abstract*. A *Concrete Number* is a number in which the kind of unit is named. An *Abstract Number* is a number in which the kind of unit is not named. A concrete number may also be defined as a number associated with something which it numbers. This is seen in the etymology of the term, *con* and *cresco*, a growing together. An abstract number may also be defined as a number not associated with anything numbered. This is indicated by the etymology of the term, *ab* and *traho*, a drawing from. It is not true, therefore, as has been asserted, that "all numbers are concrete." Number is never concrete, in the popular sense of material. When I think of *four apples*, the *apples* are concrete, but the *four* is purely numerical and in no sense material. It would be much nearer the truth to say that all numbers are abstract; for the number itself is always a pure abstraction. The distinction between an abstract and a concrete number is not a difference in the numbers themselves, but a distinction founded upon the fact of their being associated or not associated with something numbered.

This distinction is clearly seen in the origin of the idea of number. The idea of number is awakened by the contemplation of material objects. The mind takes the thought of the how-many, abstracts it from the material things with which it was at first associated, lifts it up into the region of the ideal, and conceives it as pure number. Though the idea was primarily awakened by the objects of the material world as the occasion, yet so distinct is number from matter, that if all material things were destroyed, we could still have a science of number as complete as that which now exists.

There is still another method of conceiving the distinction between concrete and abstract numbers. All numbers are composed of units. The unit gives character and value to the number of which it is the basis. A number is clearly apprehended only as we have a clear apprehension of the unit: thus, 6 *pounds* or 6 *tons* are only clear and definite ideas to us as we have clear and definite ideas of the units, *pound* and *ton*. Hence, also, the nature of numbers depends upon the nature of the units which compose them. Fundamentally, units are of two classes, concrete and abstract. A concrete unit is some object in nature or art, as, an *apple*, a *book;* or some definite quantity agreed upon to measure quantity of magnitude; as, a *yard*, a *pound*, etc. An abstract unit is merely *one* without any reference to any particular thing. The concrete unit is not a number, it is only one of the things numbered; the abstract unit is the number *one.* A collection of abstract units gives us an *Abstract Number;* a collection of concrete units gives us what is called a *Concrete Number.* An *Abstract Number* is thus merely a number of abstract units; a *Concrete Number* is a number of concrete units. The *number* itself and the *things numbered,* considered together, constitute what is called the *Concrete Number.* This is the usual method of conceiving the distinction between an abstract and a concrete number; but it is not as simple as the one previously presented.

From either method of conceiving the difference between these two classes of numbers, it will be seen that the Concrete Number is dual in its nature, consisting of two classes of units. Thus, in the concrete number, *four apples,* the concrete unit is *one apple;* while the basis of the number *four* itself is the abstract unit, *one.* Both of these classes of units must be clearly apprehended in order to have a clear and adequate idea of any concrete number.

CHAPTER IV.

NUMERICAL IDEAS OF THE ANCIENTS.

AMONG the ancients, much time was spent in discussing the properties of numbers. The science, with them, was mainly speculative, abounding in fanciful analogies. Pythagoras, the greatest mathematician of his age, was deeply imbued with this passion for the mysterious properties of numbers. He regarded number as of Divine origin, the foundation of existence, the model and archetype of things, the essence of the universe.

Plato ascribed the invention of numbers to Theuth, as may be seen in the following passage in the *Phædrus:* "I have heard, then, that at Naucratis, in Egypt, there was one of the ancient gods of that country, to whom was consecrated the bird which they call Ibis; but the name of the deity himself was Theuth. He was the first to invent numbers, and arithmetic, and geometry, and astronomy, and moreover draughts and dice, and especially letters." In the *Timæus,* he presents the conception of the relation of numbers to time, with great beauty of expression. "Hence, God ventured to form a certain movable image of eternity; and thus, while he was disposing the parts of the universe, he, out of that eternity which rests in unity, formed an eternal image on the principle of numbers, and to this we give the appellation of *Time.*"

Aristotle, in speaking of the Pythagoreans, says, "They supposed the elements of numbers to be the elements of all entities, and the whole heaven to be an harmony and number."

And again he says, "Plato affirmed the existence of numbers independent of sensibles; whereas, the Pythagoreans say that numbers constitute the things themselves, and they do not set down mathematical entities as intermediate between these."

The views of Pythagoras are so curious and interesting that they may be stated somewhat in detail. He regarded *Numbers* as of Divine origin, as above stated, and divided them into various classes, to each of which were assigned distinct properties. *Even numbers* he regarded as *feminine*, and allied to the earth; *odd numbers* were supposed to be endued with masculine virtues, and partook of the celestial nature.

One, or the *monad*, was held as the most eminently sacred, as the parent of scientific numbers. *Two*, or the *duad*, was viewed as the associate of the *monad*, and the mother of the elements, and the recipient of all things material; and *three*, or the *triad*, was regarded as perfect, being the first of the masculine numbers, comprehending the beginning, middle, and end, and hence fitted to regulate by its combinations the repetition of prayers and libations. It was the source of love and symphony, the fountain of energy and intelligence, the director of music, geometry, and astronomy. As the *monad* represented the Divinity, or Creative Power, so the *duad* was the image of matter; and the *triad*, resulting from their mutual conjunction, became the emblem of ideal forms.

Four, or the *tetrad*, was the number which Pythagoras affected to venerate the most. It is a square, and contains within itself all the musical proportions, and exhibits by summation $(1+2+3+4)$ all the digits as far as *ten*, the root of the universal scale of numeration. It marks the seasons, the elements, and the successive ages of man; and also represents the cardinal virtues, and the opposite vices. It marked the ancient fourfold division of science into arithmetic, geometry, astronomy, and music, which was termed *tetractys*, or *quaternion*. Hence, Dr. Barrow explains the oath familiar to the disciples of Pythagoras: "I swear by him who communicated the *Tetractys.*" *Five*, or the *pentad*, being composed of the first male and female numbers, was styled the number of the world. Repeated in any manner by an odd multiple, it always reappeared; and it marked the animal senses and the zones of the globe.

Six, or the *hexad*, composed of the sum of its several factors (1+2+3), was reckoned perfect and analogical. It was likewise valued as indicating the faces of the cube, and as entering into the composition of other important numbers. It was deemed harmonious, kind, and nuptial. The third power of 6, or 216, was conceived to indicate the number of years that constitute the period of metempsychosis.

Seven, or the *heptad*, formed from the junction of the *triad* and *tetrad*, has been celebrated in every age. Being unproductive, it was dedicated to the virgin Minerva, though possessed of a masculine character. It marked the series of the lunar phases, the number of the planets, and seemed to modify and pervade all nature. It was called the horn of Amalthea, and reckoned the guardian and director of the universe.

Eight, or the *octad*, being the first cube that occurred, was dedicated to Cybelé, the mother of the gods, whose image, in the remotest times, was only a cubical block of stone. From its even composition, it was termed Justice, and made to signify the highest or inerratic sphere.

Nine, or the *ennead*, was esteemed as the square of the *triad*. It denotes the number of the Muses; and, being the last of the series of digits, and terminating the tones of music, it was inscribed to Mars. Sometimes it received the appellation of Horizon, because, like the spreading ocean, it seemed to flow around the other numbers within the decad; for the same reason, it was also called Terpsichore, enlivening the productive principles in the circle of the dance.

Ten, or the *decad*, from its important office in numeration, was, perhaps, most celebrated. Having completed the cycle, and begun a new series of numbers, it was aptly called *apocatastasic*, or periodic, and therefore dedicated to the double-faced Janus, the god of the year. It had likewise the epithet of Atlas, the unwearied supporter of the world.

The cube of the *triad*, or the number *twenty-seven*, expressing the time of the moon's periodic revolution, was supposed to signify the power of the lunar circle. The quaternion of celestial numbers, *one*, *three*, *five*, and *seven*, joined to that of the terrestrial numbers, *two*, *four*, *six*, and *eight*, compose the number *thirty-six*, the square of the first perfect number, *six*, and the symbol of the universe, distinguished by wonderful properties.

In pursuit of these mystical relations and analogies, every number became, as it were, possessed of a property; and all numbers possessed some relative analogy with each other to which a name could be given. Numbers also became the symbols of intellectual and moral qualities. Thus, perfect numbers compared with those which are deficient or superabundant, are considered as the images of the virtues, regarded as equally remote from excess and defect, and constituting a mean point between them: thus, true courage is a mean between audacity and cowardice, and liberality between profusion and avarice. In other respects, also, this analogy is remarkable, as perfect numbers, like virtues, are few in number, and generated in a constant order; while superabundant and deficient numbers are like vices, infinite in number, disposable in no regular series, and generated according to no certain and invariable law.

The tracing of these analogies, accompanied, as they usually were, with moral illustrations of uncommon elegance and beauty, may be considered as furnishing a pleasing, if not a useful exercise of the understanding; but such analogies were often taken for proofs, and assumed as the bases of the most absurd and inconsistent theories. Thus Pythagoras considered "number as the ruler of forms and ideas, and the cause of gods and dæmons;" and again that "to the most ancient and all-powerful creating Deity, number was the canon, the efficient reason, the intellect also, and the most undeviating of the composition and generation of all things." Philolaus declared "that number was the governing and self-begotten bond of the eternal permanency of mundane natures." Another said, "that number was the judicial instrument of the Maker of the universe, and the first paradigm of mundane fabrication."

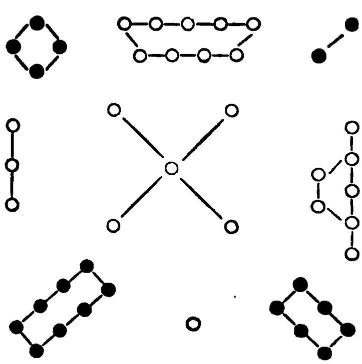
It appears to have been a favorite practice with the Greeks of the latter ages to form words in which the sum of the numbers expressed by their component letters, should be equal to some remarkable number; of this kind were the words αβρασαξ and αβρασαξα, the letters in which express numbers, which added together, are equal to 365 and 366, the number of days in the common and bissextile years respectively; and it was also remarked that the word νειλος possessed the same property as the first of these words. Words in which the sums of the numbers expressed by the letters were equal, were called ονόματα ισόψηφα; and we have an example in the Greek anthology, where a poet, wishing to express his dislike to a fellow of the name of Δαμαγορας, says, that having heard that his name was equivalent in numeral value to λοιμὸς, a pestilence, he proceeded to weigh them in a balance, when the latter was found to be the lighter.

Observations like these, however trifling, are not without their portion of curiosity; but the same indulgence cannot be shown to the absurdities of those Pythagorean philosophers, who, among other extraordinary powers which they attributed to numbers, maintained that, of two combatants, the one would conquer, the characters of whose name expressed the larger sum. It was upon this principle that they explained the relative prowess and fate of the heroes in Homer, Πατροκλος, Ἑκτωρ, and Αχιλλευς, the sums of the numbers in whose names are 871, 1225, and 1276 respectively.

This very singular superstition continued in force as late as the sixteenth century, and was transferred from the Greek to the Roman numeral letters, I, U or V, X, L, C, D, and M, which correspond to the numbers 1, 5, 10, 50, 100, 500, and 1000; thus the numeral power of the name of Maurice (Mauritius) of Saxony, was considered as an index of his success against Charles V. It was the fashion, also, to select or form memorial sentences or verses to commemorate remarkable dates. Thus the year of the Reformation (1517) was found to be expressed by the numeral letters of this verse of the *Te Deum, Tibi cherubin et seraphin incessabili voce proclamant,* in which there is one M, four C's, two L's, two U's or V's, and seven I's.

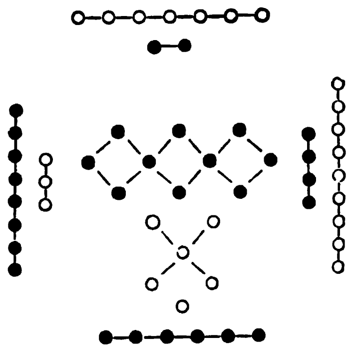
The Chinese, also, are distinguished for their arithmetical fancies. They regarded even numbers as terrestrial, and partaking of the feminine principle *Yang;* while odd numbers were regarded as of celestial extraction, and endued with the masculine principle *Y.* Even numbers were represented by small black circles; odd numbers by small white ones, variously disposed and connected by straight lines. *Thirty,* the sum of the five even numbers, 2, 4, 6, 8, and 10, was called the number of the *Earth; twenty-five,* the sum of the odd numbers, 1, 3, 5, 7, 9, and also the square of *five,* was called the number of *Heaven.*

The nine digits were grouped in two ways called *Lo-chou* and *Ho-tou.* The former expression signifies the *Book of the River Lo,* or what the Great Yu saw delineated on the back of the mysterious tortoise which rose out of that river. It may be represented as follows: *Nine* was the head, *one* the tail, *three* and *seven* its left and right shoulders, *four* and *two*



its fore feet, *eight* and *six* its hind feet. The number *five*, which represented the heart, being the square root of twenty-five, was also the emblem of Heaven. It will be noticed that this group of numbers is the common magic square of nine digits, each row of which amounts to fifteen.

The Ho-tou was what the Emperor Fou-hi observed on the body of the horse-dragon which he saw spring out of the river Ho. It consists of the first nine numbers arranged in the form of a cross. The central number was *ten*, which, it is remarked by the commentators, terminates all the operations on numbers. Other facts equally curious will be found in the literature of other nations, a full collection of which would make an interesting volume. For the facts here presented, and the manner in which they are stated, I am indebted to Leslie.



This passion for discovering the mystical properties of numbers descended from the ancients to the moderns, and numerous works have been written for the purpose of explaining them. Petrus Bungus, in 1618, wrote a work on the mysteries of numbers, extending to seven hundred quarto pages. He illustrates all the properties of numbers, whether mathematical, metaphysical, or theological; and not content with collecting all the observations of the Pythagoreans concerning them, he has referred to every passage in the Bible in which numbers are mentioned, incorporating, in a certain sense, the whole system of Christian and Pagan theology. He holds that the number 11, which *transgresses* the decad, denotes the wicked who transgress the Decalogue, whilst 12, the number of the apostles, is the proper symbol of the good and the just.

The number, however, upon which, above all others, he has dilated with peculiar industry and satisfaction, is 666, the number of the beast in Revelation, the symbol of Antichrist; and he seems particularly anxious to reduce the name of Martin Luther to a form which may express this formidable number. It may also be remarked that Luther interpreted this number to apply to the duration of Popery, and also that his friend and disciple, Stifel, the most acute and original of the early mathematicians of Germany, appears to have been seduced by these absurd speculations.

The numbers 3 and 7 were the subject of particular speculation with the writers of that age; and every department of nature, science, literature, and art, was ransacked for the purpose of discovering ternary and septenary combinations. The excellent old monk, Pacioli, the author of the first printed treatise on arithmetic, has enlarged upon the first of these numbers in a manner which is rather amusing, from the quaint and incongruous mixture of the objects which he has selected for illustration. "There are three principal sins," says he, "avarice, luxury, and pride; three sorts of satisfaction for sin,—fasting, almsgiving, and prayer; three persons offended by sin,—God, the sinner himself, and his neighbor; three witnesses in heaven,—the Father, the Word, and the Holy Spirit; three degrees of penitence,—contrition, confession, and satisfaction, which Dante has represented as the three steps of the ladder that leads to Purgatory, the first marble, the second black and rugged stone, the third red porphyry. There are three Furies in the infernal regions; three Fates,—Atropos, Lachesis, and Clotho; three theological virtues,—faith, hope, and charity; three enemies of the soul,—the world, the flesh, and the devil; three vows of the Minorite Friars,—poverty, obedience and chastity; three ways of committing sin,—with the heart, the mouth, and the act; three principal things in Paradise,—glory, riches, and justice; three things which are especially displeasing to God,—an avaricious rich man, a proud poor man, and a luxurious old man; three things which are in no esteem,—the strength of a porter, the advice of a poor man, and the beauty of a beautiful woman. And all things, in short, are founded in three, that is, in number, in weight, and in measure."

In these fanciful speculations, the number seven has received an equal, if not a greater distinction than the number three. In the year 1502, there was printed at Leipsic a work in honor of the number seven, especially composed for the use of the students of the university, which consisted of seven parts, each part consisting of seven divisions. In 1624, William Ingpen, Gent., of London, published a work entitled "The Secrets of Numbers, according to Theological, Arithmetical, Geometrical, and Harmonical Computation. Drawn for the better part, out of those ancients, as well as Neoteriques. Pleasing to read, profitable to understand, opening themselves to the capacities of both learned and unlearned, being no other than a key to lead men to any doctrinal knowledge whatsoever." Di Borgo seems to have been influenced by the same principle in determining the number of the divisions of arithmetic; for he says: "The ancient philosophers assign nine parts of algorism, but we will reduce them to seven, in reverence of the seven gifts of the Holy Spirit; namely, numeration, addition, subtraction, multiplication, division, progressions, and extraction of roots."

Some of these fancies are not entirely extinct at the present day. In England, seven constitutes the term of apprenticeship, the period for academical degrees, and as in our own country, the product of these two magic numbers *three* and *seven* constitutes the legal age of majority; and the frequent use of the number seven in the Bible has given it associations which have caused it to be regarded as a sacred number.

SECTION II.

ARITHMETICAL LANGUAGE.

CHAPTER I.

NUMERATION, OR THE NAMING OF NUMBERS.

BEGINNING at the Unit, we obtain, by a process of synthesis, arithmetical objects which we call Numbers. These objects we distinguish by names, and thus obtain the language of arithmetic. This language is both oral and written. The oral language of arithmetic is called *Numeration;* the written language of arithmetic is called *Notation.* Numeration treats of the method of naming numbers; Notation treats of the method of writing numbers. As oral language always precedes written language, it is seen that Numeration precedes Notation, and that the practice of arithmeticians in reversing this order is illogical.

Numeration is the method of naming numbers. It also includes the reading of numbers when expressed by characters. The oral language of arithmetic is based upon a principle peculiarly simple and beautiful. Instead of giving independent names to the different numbers, which would require more words even to count a million than one could acquire in a lifetime, we name a few of the first numbers, and then form groups or collections, name these groups or collections, and then use the first simple names to number the groups. The method is really that of classification, which performs for arithmetic somewhat the same service of simplification that it does in natural science. This ingenious, though simple and natural method of breaking numbers up into classes or groups, seems to have been adopted by all nations. With the civilized world and with most uncivilized tribes, these groups generally consist of ten single things, suggested, undoubtedly, by the practice among primitive races, of reckoning by counting the fingers of the two hands.

Method of Naming.—The fundamental principle of naming numbers, then, is that of grouping by tens. We regard ten single things as forming a single *collection* or *group;* ten of these groups forming a larger group, and so on; ten groups of any one value forming a new group of ten times the value, each group being regarded and used as a single thing. In this way, by giving names to the first nine numbers, and names to the groups, and employing the first nine to number the groups, we are enabled to express the largest numbers in a concise and convenient form. The value of this method of naming may be seen from the consideration that, without it, the memory would be overwhelmed by the multiplicity of disconnected words, and we should require a lifetime to learn the names of numbers, even up to a few hundred thousands. It also enables us to form a clear and distinct conception of large numbers, whose composition we discover in the words by which they are expressed, or in the symbols by which they are represented. It serves, also, as a basis for the ingenious and useful method of writing numbers, without which arithmetic would be almost useless to us.

Naming numbers in this way, a single thing is called *one;* one and one more are *two;* two and one more are *three;* and in the same manner we obtain *four, five, six, seven, eight,* and *nine,* and then adding one more and collecting them into a group, we have *ten.* Now, regarding the collection *ten* as a single thing, and proceeding according to the principle stated, we have *one and ten, two and ten, three and ten,* etc., up to *ten and ten,* which we call *two tens.* Continuing in the same manner, we have *two tens and one, two tens and two,* etc., up to *three tens,* and so on until we obtain ten of these groups of tens. These ten groups of tens we now bind together by a thread of thought, forming a new group which we call a *hun-*

dred. Proceeding from the hundred in the same way, we unite ten of these into a larger group which we name *thousand,* etc.

This is the actual method by which numbers were originally named; but unfortunately, perhaps, for the learner and for science, some of these names have been so much modified and abbreviated by the changes incident to use, that, with several of the smaller numbers at least, the principle has been so far disguised as not to be generally perceived. If, however, the ordinary language of arithmetic be carefully examined, it will be seen that the principle has been preserved, even if disguised so as not always to be immediately apparent. Instead of *one and ten* we have substituted the word *eleven,* derived from an expression formerly supposed to mean *one left after ten,* but now believed to be a contraction of the Saxon *endlefen,* or Gothic *ainlif* (*ain, one,* and *lif, ten*); and instead of *two* and *ten,* we use the expression *twelve,* formerly supposed to have been derived from an expression meaning *two left after ten,* but now regarded as arising from the Saxon *twelif,* or Gothic *tvalif* (*tva, two,* and *lif, ten.*)

With the numbers following twelve, the principle can be more readily seen, though by constant use the original expressions have been abbreviated and simplified. The stream of speech, "running day by day," has worn away a part of the primary form, and left us the words as they now exist. Thus, supposing the original expression to be *three and ten,* (originally the Anglo-Saxon *thri* and *tyn*) if we drop the conjunction *and,* we shall have *three-ten;* changing the *ten* to *teen* we have *three-teen;* then changing the *three* to *thir,* and omitting the hyphen, we have the present form *thirteen.* In a similar manner the expression *four and ten* becomes *fourteen; five and ten, fifteen; six and ten, sixteen,* etc. By the same principles of abbreviation and euphonic change, we might have obtained *twenty, thirty,* etc. Supposing the original form to be *two tens,* or *twain tens* (in the Saxon *twentig,* from *twegen,* *two*, and *tig*, *ten*), then changing the *twain* to *twen*, and the *tens* to *ty*, we shall have the common form, *twenty*. In *three tens*, changing the *three* to *thir* and the *tens* to *ty*, we have *thirty*. In the same way we obtain forty, fifty, sixty, etc., and from these by omitting the *and* in the expression *two tens and one*, *two tens and two*, etc., we have *twenty-one*, *twenty-two*, *thirty-three*, *forty-seven*, etc.

To illustrate the *law* of the formation of these names, we have used the present English forms rather than those in which the transformations actually occurred. It will be remembered that these names were derived from the Anglo-Saxon, and the changes which we have illustrated took place in that language before the names were adopted in the English tongue. The word *thirteen* was actually derived from the Anglo-Saxon *threo-tyne*, which was composed of *thri*, three, and *tyne*, ten; *fourteen* from *feowertyne*, composed of *feower*, four, and *tyne*, ten, etc. We get the word *twenty* from the Anglo-Saxon *twentig*, which is composed of the Anglo-Saxon *twegen*, two, and *tig*, ten; *thirty* from *thritig*, which is composed of *thri*, three, and *tig*, ten, etc. The law of the composition of these original words is no doubt the same as that illustrated by the use of the English words given above.

In a similar manner we name the numbers from *one hundred* to the next group, consisting of *ten hundreds*, to which we assign a new name, calling it *thousand*. After reaching the *thousand*, a change occurs in the method of grouping. Previously, ten of the old groups made one of the next higher group, but after the third group, or *thousands*, it requires a thousand of an old group to form a new group, which receives a new name. A thousand thousands forms the next group after thousands, which we call *million* from the Latin *mille*, a thousand. In the same manner, one thousand *millions* gives a new group which we call *billion*, one thousand *billions* a new group which we call *trillion*, etc.

This change in the law by which a new group is formed from an old one, is not an accident; it is intentional. It is due to science, rather than to chance. The method of counting ten in a group was commenced in an age anterior to science, and proceeded no further than hundreds and thousands, since the wants of the people did not require larger numbers; but when arithmetic began to be cultivated as a science, it was seen to be a matter of convenience to increase the size of the groups receiving a new name, and then the law became changed.

The reason that the law of naming numbers does not appear in the names of the smaller numbers, is, that they became changed from the original form on account of their frequent use. The same fact appears in grammar in the irregularity of the verbs expressing ordinary actions, as run, go, eat, drink, etc., which became thus irregular in the formation of their tenses from the constant and careless use of the common people, before the language was fixed by the rules of science or the art of printing.

Utility.—The utility of the method of naming numbers by collecting them into groups or bunches, is generally imperfectly appreciated. The method which naturally would be first suggested to the mind, is to give each number an independent name, just as we distinguish rivers, cities, states, etc. This would, of course, require a vocabulary of names as extensive as the series of natural numbers,—a vocabulary which, even for the ordinary purposes of life, could be learned only by years of labor. By the method of groups, the vocabulary is so simple that it can be acquired and employed with the greatest ease. It may be remarked, that this method of grouping, though suggested by the accidental circumstance of counting the fingers, is in accordance with that universal operation of the mind by which it binds up its knowledge into bunches or packages. It is, in fact, based upon the principle of generalization and classification.

Origin of Names.—The origin or primary meaning of the names applied to the first ten numbers, is not known. It has been supposed that the names of the simple numbers were originally derived from some concrete objects, and there are a few facts which seem to indicate the correctness of this supposition. Thus, the Persian name for *five* is *pendje*, while *péntcha* means the *expanded hand*, and the corresponding terms in the Sanskrit are said to have a similar meaning. The term *linia*, which with slight modifications is used for *five* throughout the Indian Archipelago, means *hand* in the language of the Otaheite and other islands. Among the Jaloffs, an African tribe, the word for *five*, *juorum*, likewise signifies *hand*. Among the Greenlanders the term for *twenty* is *innuk*, or *man*; that is, after completing the counting of fingers and toes, they say *innuk* or *man;* and there are also examples of the identity of the term for *man* and *twenty* among some of the tribes of South America.

Among the Indians of Bogota, New Grenada, the term *quicha*, meaning a *foot*, is used to number the second decade, while *twenty* is named *gueta*, which signifies a *house*. Nearly all the South American tribes use the word for *hand* to express *five*, and in many cases the word for *man* is used to express *twenty*. A tribe in Paraguay denote *four* by an expression which means the *fingers* of the *Emu*, a bird common in Paraguay, possessing *four* claws on each foot, *three* before, and *one* turned back; and their word for *five* is the name of a beautiful skin with *five* different colors. The same number is, however, more commonly expressed by *hanam begem*, the *fingers of one hand; ten* is expressed by *the fingers of both hands;* and for *twenty* they say *hanam rihegem cat grachahaka anomichera hegem, the fingers of both hands and feet.* Among the Caribbeans, the fingers are termed the children of the hand, and the toes children of the feet; and the phrase for *ten*, *chou oucabo raim*, means *all the children of the hands.*

Humboldt has given from the researches of Duquesne, the etymological signification of some of the numerals of the Indians of New Grenada. Thus, *ata*, one, signifies *water;*

bosa, two, an *enclosure; mica*, three, *changeable; muyhica*, four, *a cloud threatening a tempest; hisa*, five, *repose; ta*, six, *harvest; cahupqua*, seven, *deaf; suhuzza*, eight, *a tail;* and *ubchica*, ten, *resplendent moon*. No meaning has been discovered for *aca*, the numeral for *nine*. It would seem impossible, amidst such various meanings, to discover any principle which may seem to have pointed out the use of these terms as numerals.

In the Mexican numeral symbols there is an intelligible connection between the sign and the thing signified, though the association seems to be entirely arbitrary. Thus, the symbol for *one* is *a frog;* for *two, a nose with extended nostrils*, part of the lunar disk, figured as a face; for *three, two eyes open*, another part of the lunar disk; for *four, two eyes closed;* for *five, two figures united*, the nuptials of the sun and moon, conjunction; for *six, a stake with a cord*, alluding to the sacrifice of Guesa tied to a pillar; for *seven, two ears;* for *eight*, no meaning assigned; for *nine, two frogs coupled;* for *ten, an ear;* for *twenty, a frog extended.*

The following theory, advanced by Prof. Goldstücker, in a paper read before the Philological Society in 1870, in which he gives good linguistic evidence in support of the origin of the Sanskrit numerals, and consequently of our own, is at least plausible, and will be interesting: *One*, he says, is "he," the third personal pronoun; *two*, "diversity;" *three*, "that which goes beyond;" *four*, "and three," that is, "one and three;" *five*, "coming after;" *six*, "four," that is, "and four," or "two and four;" *seven*, "following;" *eight*, "two fours," or "twice four;" *nine*, "that which comes after" (ch. *nava*, new); *ten*, "two and eight." Thus, only *one* and *two* have distinct original meanings. After giving these, our ancestors' powers needed a rest; then they made *three*, and added to it *one* for *four;* then took another rest, repeated the notion of *three* in *five*, and the notion of *four* in *six;* then rested once more, and again repeated the notion of *three* and *five* in *seven;* took another rest, and got a new idea of two *fours* for *eight;* but for *nine* repeated for the fourth time the "coming after" notion of *three, five,* and *seven;* while for *ten* they repeated for the third time the addition-notion of *four* and *six.* The Professor insists strongly on this seeming poverty and helplessness of the early Indo-European mind. He does not put forward the above meanings of the numerals as new, though he believes that his history of most of the forms of their names is so. The anomalous form of the Sanskrit *shash,* six—the hardest of them—first set him at work on the numerals, and the Zend form *kshvas* led him to the true explanation of this, and thence to that of the other numerals.

In closing this chapter, we remark that the names of the periods above duodecillions have not been fully settled by usage. Prof. Henkle, who has examined the subject with considerable care, finds a law which he maintains should hold in the formation of the names of the higher periods. The terms *quintillions, sextillions,* and *nonillions* are formed, not from the cardinals, *quinque, sex,* and *novem,* but from the ordinals, *quintus, sextus,* and *nonus.* From this he infers that analogy plainly demands that the names beyond *duodecillions* should be formed from the Latin ordinal numerals. For the names thus formed, see appendix.

CHAPTER II.

NOTATION, OR THE WRITING OF NUMBERS.

ARITHMETICAL language is the expression of arithmetical ideas. These ideas may be expressed in sound to the ear, or in visible form to the eye; arithmetical language is, therefore, both oral and written. The oral language is called *Numeration;* the written language, *Notation.* Numeration is the method of naming numbers; Notation is the method of writing numbers. From this consideration it would seem that the written language of arithmetic must bear an intimate relation to the oral language, which we find to be the case. The general method of writing numbers, now adopted by all civilized nations, is the Hindoo, usually called the Arabic method. This method is based upon, and arises naturally out of, the method of naming numbers by groups.

The fundamental principle of the Arabic system is the ingenious and refined idea of *place value.* Recognizing the method of naming numbers by groups, it assumes to represent these groups by the *simple device of place.* It fixes upon a few characters to represent a few of the first numbers, and then employs these same characters to number the groups, the group numbered being indicated by the place of the character. This leads to the distinction of the *intrinsic* and *local* value of the numerical characters. Each character has a definite value when it stands alone, and a relative value when used in connection with other characters.

The number of the arithmetical characters is determined by the number of units in the group. The grouping being by tens, the number of characters needed is only nine, one less than the number of units in the group. These characters are called digits, from the Latin *digitus*, a finger, the name commemorating the ancient custom of reckoning by counting the fingers. In the combination of these characters to express numbers, it will often be required to indicate the absence of some group; hence arises the necessity of a character which expresses no value, a character which denotes merely the absence of value. This character is known as *naught*, or *zero*. We thus have the following ten characters: 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, with which we are able to express all possible numbers.

Utility.—The Arabic system, based upon the refined idea of place value, is one of the happiest results of human intelligence, and deserves our highest admiration. Remarkable as is its simplicity, it constitutes, regarded in its philosophical character or its practical value, one of the greatest achievements of the human mind. In the hands of a skillful analyst, it becomes a most powerful instrument in wresting from nature her hidden truths and occult laws. Without it, many of the arts would never have been dreamed of, and astronomy would have been still in its cradle. With it, man becomes armed with prophetic power,—predicting eclipses, pointing out new planets which the eye of the telescope had not seen, assigning orbits to the erratic wanderers of space, and even estimating the ages that have passed since the universe thrilled with the sublime utterance, "Let there be light!" Familiarity with it from childhood detracts from our appreciation of its philosophical beauty and its great practical importance. Deprived of it for a short time, and compelled to work with the inconvenient methods of other systems, we should be able to form a truer idea of the advantages which this invention has conferred on mankind.

Relation to Numeration.—Though the methods of notation and numeration are intimately related, there is also an essential distinction between them. Though similar, they are by no means identical in principle. Their similarity is seen in the fact that the method of notation could not be applied without the method of numbering by groups; their distinction is seen in the fact that we could have the present method of numeration without the Arabic system of notation. The notation seems to be an immediate outgrowth from the numeration, yet not a necessary one; for many nations who had the same method of naming numbers, employed other methods of writing them.

Their true relation also appears in considering their common relation to the decimal scale. The decimal principle belongs both to our method of naming and of writing numbers. This coincidence is not accidental, but essential to the harmony of oral and written expression. The necessity of this would be very apparent if we should attempt to change the base of the scale of notation without changing the base of the method of naming numbers. With our present base we say *one and ten*, *two and ten*, etc., or at least their equivalents; and our written expressions are read in the same manner. Should we adopt any other scale of notation, retaining our present base in naming numbers, the reading of numbers in this new scale would be so awkward and inconvenient as to be almost impossible. Hence it follows, that for a scale of notation to be advantageously employed, the methods of naming and writing numbers should possess the same basis. Thus, if the scale of notation be *quinary*, instead of naming numbers *five*, *six*, *seven*, etc., we should say *five*, *one and five*, *two and five*, etc.; if the scale were *senary*, we should say *six*, *one and six*, *two and six*, etc.

Relation to the Base.—It will also be seen that the principle of the methods of naming and writing numbers is entirely distinct from the number used as the base. The intimate association of the Arabic system with the base, has sometimes led to the idea that the base is a part of the system itself. This error should be carefully avoided. The Arabic method assumes that we name numbers by groups, and that each group contains ten; but it is in principle entirely independent of the number constituting a group. The number in the group determines the base of the scale, and consequently the number of characters to be used, but does not affect the principle of the method, which is simply that of *place value.* Should we change the base of numbering, it would change the *names* of the numbers after twelve, and the *base* of the Arabic scale; but it would in no wise affect the principle of either the method of numeration or of notation.

Number of Characters.—The number of characters in the Arabic system of notation depends upon the number of units in the groups of numeration. Thus, we must have as many simple characters as will express the different numbers from *one* until we reach within a unit of the group. We shall have no character for the group, since, according to the device of *place value*, it is to be indicated by changing the place of the symbol which represents *one*, it being *one* of the *first group.* The number of significant characters must, therefore, be always one less than the number denoting the base of the system. In the decimal scale the number of digits is *nine;* in an *octary* scale it would be *seven;* in a *quinary* scale, *four*, etc.

Origin.—The origin of this system of notation is now universally accredited to the Hindoos. When, by whom, and how it was invented, we do not know. It is not improbable that it began with the representation of the spoken words by marks, or abstract characters. They may at first have given independent characters to the numbers as far as represented. It then probably occurred to them that, since they gave independent names to a few numbers and then numbered by groups, they could simplify their system of notation by making it correspond to their system of numeration. Then first dawned upon the mind the idea of a few characters to represent the first simple numbers, and the use of these same characters to number the groups. They now stood on the threshold of one of the greatest discoveries of all time. Here arose the question—How are these groups to be distinguished? How shall we determine when a character denotes a number of *units* or *tens*, or *hundreds*, etc.? How many methods occurred to them before the method of *place*, who can tell? This might have been done by slightly varying the character, by attaching some mark to it, by annexing the initial of the group, etc.; either of which would have been comparatively complicated and inconvenient. At last, to the mind of some great thinker, occurred the simple idea of *place value*, and the problem was solved. "Who was the man?" is a question answered only by its own echo, for his name sleeps in the silence of the past. Were it known, mankind would feel like rearing a monument to his memory, as high and enduring as the Pyramids of Egypt; but now it can only raise its altar to the Unknown Genius.

Origin of Characters.—The origin of the characters, like that of the system, is shrouded in mystery; not a ray of light upon the subject comes down the historic path. Many of the early writers gave some ingenious speculations concerning their origin. Gatterer imagined that he had discovered in Egyptian manuscripts written in the enchoriac character, that the digits were denoted by nine letters; and Wachter supposed them to have a natural origin in the different combinations of the fingers: thus, unity is expressed by the outstretched finger; two by two fingers, which may have been represented by two marks that, by long use, passed into the present form, and so on for all the other symbols. In the absence of facts, three theories have been presented, which are at least interesting on account of their ingenuity, and are certainly somewhat plausible. One of these theories is that they are formed by the combination of straight *lines*, as the primary representation of numbers; another is that they are formed by the combination and modification of *angles;* and still another and more recent theory is that they are the *initial letters* of the Hindoo numerals. These three theories may be distinguished as the theories of *lines*, *angles*, and *initial letters*.

The first theory is based on the primary use of straight lines to represent numbers. By this method, one straight line, |, would represent *one;* two straight lines which may have been connected thus, L, *two;* three lines, thus, >, or with a connecting curve, thus, 3, *three;* four lines arranged thus, □, or thus, 4, *four;* five lines arranged thus, 5, *five;* six lines arranged thus, 6, *six;* seven lines, thus, 9, *seven;* eight lines thus, 8, or thus 8, *eight;* nine lines, thus, 9, *nine.* The *zero* is supposed to have been originally a circle, suggested from counting around the fingers and thumbs held in a circular position.

The second theory is based upon the use of angles to represent numbers. The ancient mathematicians were noted for their astronomical observations and calculations, and being thus familiar with the use of angles, it is not unreasonable to suppose that they would employ the angle in their representation of numbers. Thus, they might very naturally have used one angle, 1, for *one;* two angles, Z, for *two;* three angles 3, for *three;* four angles, 4, for *four;* five angles, 5, for *five,* six angles, 6, for *six;* seven angles, 7, for *seven;* eight angles, 8, for *eight;* nine angles, 9, for *nine.* These characters being frequently made, would eventually assume the rounded form which they now possess. By this theory, the character for zero is easily and naturally accounted for If angles were used to represent numbers, *nothing* would be represented by a character having *no angles,* which is the closed curve.

The latest and most plausible theory for the origin of Arabic characters is, that they were originally the initial letters of the Sanskrit numerals. This theory is presented by Prinseps, a profound Sanskrit scholar, and is indorsed by Max Müller. Such a use of initial letters was entirely feasible in the Sanskrit language, as each numeral began with a different letter. The plausibility of the theory further appears from the fact that it follows the general law of representing numbers by letters, as in the Roman, Greek, and Hebrew systems.

This theory does not account for the origin of the zero, the most important character of them all,—in fact, the key to the system of modern arithmetic. No other system of notation except the sexagesimal system, had it. Max Müller says: "It would be highly important to find out at what time the naught first occurs in Indian inscriptions. That inscription would deserve to be preserved among the most valuable monuments of antiquity, for from it would date in reality the beginning of true mathematical science—impossible without the naught—nay, the beginning of all the exact sciences to which we owe the invention of telescopes, steam engines, and electric telegraphs." Dr. Peacock supposes that it was derived from the Greek ο, introduced by Ptolemy to denote the vacant places in the sexagesimal arithmetic; the Hindoos, he says, having used a dot for this purpose.

It seems to have been difficult at first to comprehend the precise force of the *cipher*, which, insignificant in itself, serves only to determine the rank and value of the other figures. When they were first introduced into Europe, it was deemed necessary to prefix to any work in which they were used, a short treatise on their nature and application. These notices are often met with attached to old vellum almanacs, or inserted in the blank leaves of missals, and frequently intermixed with famous prophecies, most direful prodigies, and infallible remedies for scalds and burns. A sort of mystery, which has imprinted its trace on our language, seemed to hang over the practice of using the cipher; and we still speak of *deciphering* and writing in *cipher*, in allusion to some dark or concealed art. Indeed, in the early history of arithmetic in Europe, either on account of its association with the infidel Mohammedans from whom it was derived, or of the popular prejudice against learning which prevailed at that time, the system was regarded as belonging to black art and the devil; and it was, no doubt, this popular prejudice that delayed its general introduction into Christian Europe.

CHAPTER III.

ORIGIN OF ARITHMETICAL SYMBOLS.

THE symbols of arithmetic may be divided into three general classes: Symbols of Number, Symbols of Operation, and Symbols of Relation. What is the origin of these symbols; who invented them, or first employed them? This question, a very interesting one, I shall endeavor to answer in the present chapter.

I. SYMBOLS OF NUMBER.—The *Symbols of Number* employed by different nations, are the Arabic figures and the letters of the alphabet. Nearly all civilized nations seem to have made use of the letters of the alphabet to represent numbers. The Greeks divided their letters into several classes, to represent the different groups of the arithmetical scale. The Roman system employed the seven letters, I, V, X, L, C, D, and M, to represent numbers. The Arabs at first used the Greek method, and afterward exchanged it for that of the Hindoos.

There are three theories given for the origin of the Arabic symbols of notation, known respectively as the theory of *lines*, of *angles*, and of *initial letters*. These three theories are explained in the chapter on Notation. It may also be remarked that some of the Arabian authors who treat of astrological signs, allege that the Indian or Arabic numerals were derived from the quartering of the circle, and Leslie says that the resemblance of these natural marks to the derivative ones appears very striking. The Roman symbols are supposed to have originated in the use of simple straight lines or strokes, variously combined, for which were subsequently substituted the letters of the alphabet. This theory is explained at length in the chapter on Roman Notation.

II. Symbols of Operation.—The *Symbols of Operation* are the signs of *addition, subtraction, multiplication, division, involution, evolution,* and *aggregation.* The origin of most of these symbols has been definitely determined.

The *Symbols of Addition and Subtraction,* (+) and (—), were first introduced by Michael Stifel, a German mathematician of the sixteenth century. They first appeared in a work published by him at Nuremberg, in 1544, and are believed to have been invented by him. This is implied by the manner in which he introduces them: "thus, *we place* this sign," etc., and "*we say* that the addition is thus completed," etc. Prof. Rigaud supposed that + was a corruption of P, the initial of *plus*, and Dr. Davis thought that it was a corruption of *et* or *&*. Stifel, however, does not call the signs *plus* and *minus*, but *signum additorum* and *signum subtractorum*, which renders these suppositions improbable.

Dr. Ritchie suggested that perhaps + was two marks joined together, to signify two numbers joined together in addition; and that — was taken to indicate subtraction, since it is *what is left* after one of the marks is removed. It is thought by De Morgan that the minus sign (—) was first used, and that + was derived from it by putting a small cross-bar for distinction. "The sign +," he says, "in the hands of Stifel's printer, has the vertical bar much shorter than the other; and when it is introduced into the wood-cuts of the engraver, the disproportion is greater still." The Hindoos, from whom our knowledge of algebra was originally derived, used a dot for subtraction, and the absence of the dot for addition. It is not unlikely that the Hindoo dot was elongated into a bar to signify subtraction, and that the first who found it convenient to introduce a sign for addition, merely adopted the sign for subtraction with a difference.

Some have supposed that Stifel might have obtained these symbols from some other mathematician of his age; but this is improbable. The person to whom he refers as his principal teacher in algebra, was Christoffer Rudolph, who published, in 1561, a work entitled, *Kunstliche Rechnung mit der ziffer und mit der zalpfenningen;* but there is nothing in this work like either of the signs + or —, so that it appears quite certain that Stifel did not obtain them from him.

M. Libri attributes the invention of + and — to Leonardo da Vinci, the celebrated Italian artist and philosopher; other writers, however, say that Da Vinci used the symbol + for the figure 4. After the most careful investigation, the invention and introduction of these two symbols are almost universally accredited to Stifel. It may be remarked that these symbols were not immediately adopted by other mathematicians. In a work on algebra, published in 1619, the signs of addition and subtraction are P and M, with strokes drawn through them,

The *Symbol of Multiplication* (×), St. Andrew's cross, was introduced by William Oughtred, an eminent English mathematician and divine, born at Eton in 1573. The work in which this symbol first appeared was entitled *Clavis Mathematicæ*, "Key of Mathematics," and published in 1631. Oughtred was a fine thinker, and was honored by the title "prince of mathematicians."

The *Symbol of Division* (÷) was invented by Dr. John Pell, Professor of Philosophy and Mathematics at Breda. He was born at Southwick in Sussex, 1610, and died in 1685. This symbol was used by some old English writers to denote the ratio or relation of quantities. I have also noticed it used thus in some old German mathematical works. Dr. Pell was highly regarded as a mathematician. It was to him that Newton first explained his invention of fluxions.

The *System of Exponents*, to represent the powers of a number, was introduced by Descartes, the illustrious metaphysician and inventor of Analytical Geometry. The earliest writer on algebra denoted the powers of a number by an abbreviation of the name of the power. Harriot, a mathematician of the 17th century, repeated the quantity to indicate the power; thus, for a^4 he wrote *aaaa*.

The *Radical Sign* ($\surd$) was introduced by Stifel, the inventor of + and —. This symbol is a modification of the letter *r*, the initial of *radix*, root. The root of a quantity was formerly denoted by writing the letter *r* before it, and this letter was gradually changed to the form $\surd$.

The *Vinculum* or *Bar*, placed over quantities to connect them together, thus, $4 \times \overline{3+5}$ was first used by Vieta, the introducer, in algebra, of the system of representing known quantities by symbols. The *Parenthesis* was first used by Albert Girarde, a Dutch writer on algebra, of the sixteenth century. Who first introduced the other signs of aggregation I have not been able to ascertain.

III. SYMBOLS OF RELATION.—*Symbols of Relation* are the signs of *equality*, *ratio*, *equal ratios*, *inequality*, and *deduction*. The origin of a few of these has been ascertained.

The *Symbol of Equality* (=) was introduced by Robert Recorde, an English physician and mathematician of the sixteenth century. It first appeared in his work on algebra, called by the singular title, *Whetstone of Wit*. He gives his reason for the symbol in the following quaint language: "And to avoide the tediouse repetition of these words, I will settle as I doe often in worke use, a pair of parallels or Gemowe lines of one lengthe, thus: =, because no 2 thynges can be more equalle."

This sign was also employed by Albert Girarde. The French and German mathematicians used the symbol ∞ to denote equality, even long after Recorde. This symbol is said to be a modification of the diphthong æ, the initial of the Latin phrase *æquale est*.

The *Symbol of Ratio* (:) is supposed to be a modification of the sign of division. The sign of division was frequently employed by the old English and German mathematicians to indicate the relation of quantities. Who first omitted the dash and employed the present form of the symbol of ratio, I have not been able to ascertain.

The *Symbol of Equal Ratios* (::) may be a modification of the sign of equality (=) or a duplication of the symbol of ratio (:), but this is not certain. It seems to have been introduced by Oughtred, in a work published in 1631.

The *Symbols of Inequality* ($>$ and $<$) are evidently modifications of the sign of equality. If parallel lines denote equality, oblique lines would naturally be used to denote inequality, the lines converging towards the less quantity. Who first employed this sign is probably known, but I have no note of it, and have nothing upon it in my library.

I have now presented, in a connected and systematic manner, about all that is known concerning the origin of the ordinary arithmetical symbols.

CHAPTER IV.

THE BASIS OF THE SCALE OF NUMERATION.

THE Basis of our scale of numeration and notation is decimal. This basis is not essential, but accidental. Mankind commenced reckoning by counting the fingers of the left hand, including the thumb, and thus at first probably reckoned by *fives*. As the art of numbering advanced, they adopted a group, derived from the fingers of both hands, and thus *ten* became the basis of numbering. The decimal base was consequently determined by the number of fingers on each hand. Had there been three fingers and a thumb, the scale would have been octary; had there been five fingers and a thumb, the scale would have been duodecimal, which would have been a great advantage to arithmetic, whatever it might have been to the hand itself.

The universal use among civilized nations of the decimal scale of numeration seems to imply some peculiar excellence in it. It appears as if nature had pointed directly to it, on account of some essential fitness of the number ten, as the numerical basis. Indeed, this opinion has been quite general, and the habit acquired from the use of the system has served to confirm the belief. Many persons get the *base* of numeration and the *mode* of notation so mingled together that they see in the Arabic system nothing save the decimal basis of numeration, and attribute to it all those high qualities which belong to the mode only. It is this which has led some persons to regard the decimal basis as the perfection of simplicity and utility.

A little reflection, however, will prove that such an assumption is groundless. Although the decimal scale has been adopted by every civilized nation, yet, as has been shown, the selection was accidental, and the base entirely arbitrary. The selection occurred before attention was given to a general system, in short, without reflection, and its supposed perfection is a mere delusion. Any other number might have been taken as the root of the numerical scale; and, were a new basis to be selected by mathematicians familiar with the properties of numbers, there are several considerations that would lead them to adopt some other basis than the decimal. Some of the objections to the decimal basis will be stated, and a few considerations presented in favor of some other number as the basis of the language of arithmetic.

First, the decimal scale is unnatural. It has been superficially urged that the decimal scale is the most natural one that could have been selected. On the contrary, there is nothing natural about it, except the fingers, and a little reflection would have shown that these are grouped by *fours* instead of *fives*. In fact, a group by tens is seldom seen, either in nature or in art. What things exist by tens, associate by tens, or separate into tenths? Nature groups in pairs, in threes, in fours, in fives, and in sixes; but seldom, if ever, in tens. Man doubles and triples and quadruples his units; he divides them into halves and thirds and quarters; but where does he estimate by tens or tenths? It is thus seen that the grouping by tens is an unnatural method, suggested neither by nature nor the practical requirements of art.

Second, the decimal scale is unscientific. The confused idea of the relation of the base of the scale to the mode of notation, has led some to suppose that the decimal scale is one of the triumphs of science. The truth is, as has already been shown, that not only was it not established upon scientific principles, but it is really a violation of those principles. The decimal scale originated by chance, by a mere accident Men had ten fingers, including the thumbs, and found it convenient to reckon by counting their fingers; and thus acquired the habit of counting by tens. Had science, instead of chance, presided at its birth, we should have a basis that would have given a new beauty and a greater simplicity to our already admirable system of arithmetical language.

Third, the decimal scale is also inconvenient. It has been held not only that the decimal basis is scientific, but that it is the most convenient one that could have been selected. It needs but little reflection to see the incorrectness of this assumption. One essential for the basis of a scale is the property of its being divisible into a number of simple parts, so that it may be a multiple of several of the smaller numbers. The number *ten* will admit of only two such divisions, the *half* and the *fifth*. The *third*, *fourth*, and *sixth* are not exact parts of the denary base, in consequence of which it is inconvenient to express these fractions in the scale. Were the basis *twelve* instead of ten, we could obtain the *half*, *third*, *fourth*, and *sixth*, and these fractions could be expressed by the scale in a *single place;* whereas the *fourth* now requires two places (.25), and the *third* and *sixth* cannot be expressed exactly in a decimal scale, except as a circulate.

Essentials of a Base.—It will be interesting to notice some of the essentials of a base, and to observe what number complies most fully with these requirements. The first essential of a good base is that it will admit of being divided into the simple fractional parts; the second is that the number be neither too large nor too small. The advantage of the capability of being divided into simple fractional parts is that such fractions may be readily expressed in the terms of the scale as we now express decimal fractions. In the decimal scale only *one-half* and *one-fifth* can be expressed in one place of decimals, since they are the only exact parts of ten. With a scale whose basis is a multiple of *two, three, four* and *six*, each corresponding fraction could be expressed in terms of the scale in a single place.

In respect to the size of the base, if the number is too small, it will require too many names and places to express large numbers. If the number is very large, it will group together too many units to be apprehended and easily used in numerical operations.

Other Scales.—There are several other bases which have been recommended as preferable to the decimal; the most important of which are the *Binary*, the *Octary*, and the *Duodecimal.* The Binary scale was proposed and strongly advocated by Leibnitz. He maintained that it was the most natural method of counting, and that it presented great practical and scientific advantages. He even constructed an arithmetic upon this basis, called *Binary Arithmetic.* The obvious objection to this base is, that it would require too many names and too many places in writing large numbers. The Octary system has also been strongly advocated. A very able article in an American journal says that the binary base is the only proper base for gradation, and the octary is the true commercial base of numeration and notation.

It is probable, however, taking all things into consideration, that the *duodecimal* scale would be the most suitable. The number *twelve* is neither too large nor too small for convenience. Its susceptibility of division into halves, thirds, fourths, and sixths, is an especial recommendation to it. So great are these advantages, that, if the base were to be changed, the duodecimal base would, without doubt, be selected.

The advantage of the duodecimal scale is especially apparent in the expression of fractions in a form similar to our decimal fractions. In the decimal scale, $\frac{1}{2}$ and $\frac{1}{5}$ are the only simple fractions that can be expressed by the scale in a single place; $\frac{1}{3}$ cannot be expressed at all as a simple decimal; $\frac{1}{4}$ requires two places, and $\frac{1}{6}$, like $\frac{1}{3}$, gives an interminate decimal. With a duodecimal scale we could express $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ in a single place; while $\frac{1}{8}$ and $\frac{1}{9}$ would require only two places. Thus, in the duodecimal scale, we should have $\frac{1}{2}=.6$; $\frac{1}{3}=.4$;

$\frac{1}{4}=.3$; $\frac{1}{6}=.2$; $\frac{1}{8}=.16$, and $\frac{1}{9}=.14$. This is a very great simplification; and since all combinations of 2 and 3 could be readily expressed, and since these constitute such a large proportion of numbers, it is evident that the simplification of the subject, by means of a duodecimal scale, would be very considerable.

I will arrange the expressions of these fractions in the decimal and duodecimal scales, side by side, that the advantage of the latter may be more clearly seen.

Decimal Scale.		Duodecimal Scale.	
$\frac{1}{2}=.5$	$\frac{1}{6}=.166+$	$\frac{1}{2}=.6$	$\frac{1}{6}=.2$
$\frac{1}{3}=.333+$	$\frac{1}{7}=.\dot{1}4285\dot{7}$	$\frac{1}{3}=.4$	$\frac{1}{7}=.\dot{1}86\Phi3\dot{5}$
$\frac{1}{4}=.25$	$\frac{1}{8}=.125$	$\frac{1}{4}=.3$	$\frac{1}{8}=.16$
$\frac{1}{5}=.2$	$\frac{1}{9}=.111+$	$\frac{1}{5}=.\dot{2}49\dot{7}$	$\frac{1}{9}=.14$

It will be seen that in the decimal scale all the simple fractions used in practice, except $\frac{1}{2}$, give circulates or require two or three figures to express them; while in the duodecimal scale all the fractions ordinarily used in business transactions are expressed in a single place, and even $\frac{1}{8}$ and $\frac{1}{9}$ require only two places. The fractions $\frac{1}{5}$ and $\frac{1}{7}$ cannot be exactly expressed in the scale, but these fractions are seldom used in business. It will be interesting to notice that $\frac{1}{5}$ and $\frac{1}{7}$ both give *perfect repetends* in the duodecimal scale, and that they possess the same properties as perfect repetends in the decimal scale.

There seems to have been a natural tendency towards a duodecimal scale. Thus, a large number of things are reckoned by the *dozen*, and this scale is even extended to the *gross* and the *great-gross;* that is, to the second and the third powers of the base. Again, in our naming of numbers, the terms *eleven* and *twelve* seem to postpone the forming of a group until we reach a dozen. A similar fact appears in extending the multiplication table to include *twelve times*, since, with the decimal scale, it could conveniently stop with *nine* or *ten times*. The division of the year into twelve months, the circle into twelve signs, the foot into twelve inches, the pound into twelve ounces, etc., are each a further indication of the same tendency.

Change of Base.—The objections to the decimal base have led scientific men to advocate a change in our scale of numeration and notation. Such a change would, without doubt, be a great advantage, both to science and to art; yet the practical difficulties attending such a change are so great that it seems to be almost impossible. A change in the base would require a complete change in the oral language of arithmetic. The decimal scale is so interwoven with the speech of nations, that such a change could be effected only after years of labor. For a while, it would be necessary to have two methods of arithmetic taught and in use, as in Europe at the time of the transition from the Roman to the Arabic system of notation. The learned would soon adopt the new method, but the common people would cling with such tenacity to the old, that even a century might intervene before the new method would become generally established.

Will this change ever be made? is a question which is sometimes asked. I do not know; but I am strongly in favor of it, and believe it possible. The diffusion of popular education will prepare the way for it, by removing the difficulties of its adoption. These difficulties, though great, are not insurmountable. Changes of notation have taken place in several different nations, and some nations have changed two or three times. The Greeks changed theirs, first for the alphabetic, and afterwards, with the rest of the civilized world, for the Arabic system. The Arabs themselves first adopted the Greek, and afterwards changed it for the Hindoo method. The people of Europe changed from the Roman to the Arabic system even as late as the fourteenth century, though it took one or two centuries to effect the transition. What was done thus early in the history of science, could, with the increased intelligence of our people, be much more readily accomplished at the present day. A writer in one of our American periodicals says: "The probability is that it will be done. The question is one of time rather than of fact, and there is plenty of time. The diffusion of education will ultimately cause it to be demanded."

It is a curious fact, and one worthy of remembrance, that Charles XII. of Sweden, a short time before his death, while lying in the trenches before the Norwegian fortress of Frederickshall, seriously deliberated on a scheme of introducing the duodecimal system of numeration into his dominions.

CHAPTER V.

OTHER SCALES OF NUMERATION.

AS we have seen, any number might have been taken as the basis of the scale of numeration, the number ten, the basis of our present scale, being selected from the circumstance of there being ten fingers on the two hands. Some other scales have actually existed, and it will be interesting to notice, in various languages, traces of an earlier and simpler mode of reckoning. In order to a clearer notion of the subject, it may be premised that a scale whose basis is two is called *Binary;* three, *Ternary;* four, *Quaternary;* five, *Quinary;* six, *Senary;* seven, *Septenary;* eight, *Octary;* nine, *Nonary;* ten, *Denary;* twelve, *Duodenary*, etc.

The earliest method of numeration was that of combining units in *pairs*. It is still familiar among sportsmen, who reckon by *braces* or *couples*. Some feeble traces of the Binary system are found in the early monuments of China. *Fouhi*, the founder and first emperor of that vast monarchy, is venerated in the East as a promoter of geometry and the inventor of a science, the knowledge of which has been lost. The emblem of this occult science appears to consist of eight separate clusters of three parallel lines or *trigrams*, drawn one above the other after the Chinese manner of writing, and represented either as entire or broken in the middle. These varied trigrams were called *Koua* or *suspended symbols*, from the custom of hanging them up in the public places. In the formation of such clusters, we may perceive the application of the binary scale as far as three ranks, or the number *eight*. The entire lines are supposed to signify *one, two,* or *four*, according to their order, while the broken lines are valueless, and serve merely to indicate the rank of the others. If this be true, it furnishes an example of a species of arithmetic with the device of place, possessing an antiquity of more than 3000 years.

The Binary scale, though never fully adopted by any nation as a method of counting, has been recommended by one of the most celebrated modern philosophers, Leibnitz, as presenting many advantages, from its enabling us to perform all the operations in symbolic arithmetic by mere addition and subtraction. Such a system would, of course, require but two symbols, unity and zero, by means of which all numbers could be expressed. Thus, *two* would be expressed by 10, *three* by 11, *four* by 100, *five* by 101, *six* by 110, *seven* by 111, *eight* by 1000, etc. This system was studiously circulated by its author by means of scientific journals and his extensive correspondence; and was communicated by him to Bouvet, a Jesuit missionary at Pekin, at that time engaged in the study of Chinese ambiguities, and who imagined that he had discovered in it a key to the explanation of the Cova, or *lineations* previously referred to.

This system was also recommended by the theological idea associated with it, of which it was claimed to be the representative. As unity was considered the symbol of Deity, the formation of all numbers out of zero and unity was considered, in that age of metaphysical dreaming, as an apt image of the creation of the world, by God, from chaos. It was with reference to this view of the binary arithmetic, that a medal was struck, bearing on its obverse, as an inscription, the Pythagorean distich, *Numero Deus impari gaudet*, and on its reverse the appropriate verse descriptive of the system which it celebrated, *Omnibus ex nihilo ducendis sufficit Unum.* The good Jesuit, who seemed to have caught the spirit of Chinese belief, regarded the Cova, which were supposed to conceal great mysteries, as the symbols of binary arithmetic, as a most mysterious testimony to the unity of the Deity, and as containing within them the germ of all the sciences.

To count by *threes* was another step, and this has been preserved by sportsmen under the term *leash*, meaning the strings by which *three* dogs, and no more, can be held at once in the hand. The numbering by *fours* has had a more extensive application; it was evidently suggested by the custom of taking, in the rapid counting of objects, a pair in each hand, and thus reckoning by *fours*. English fishermen, who generally count in this way, call every *double pair* (of herring, for instance), a *throw* or *cast;* and the term *warp*, which originally meant to *throw*, is employed to denote *four*, in various articles of trade. It is alleged that the Guaranis and Sulos, two of the lowest races of savages inhabiting the forests of South America, count only by *fours;* at least they express the number *five* by *four* and *one*, *six* by *four* and *two*, *seven* by *four* and *three*, etc. It has been inferred, also, from a passage in Aristotle, that a certain tribe of Thracians were accustomed to use the quaternary scale of numeration.

The *Quinary* system, which reckons by *fives*, or *pentads*, has its foundation in the practice of counting the fingers of one hand. It appears, from the statements of travellers, to have been adopted by various savage nations. Thus, certain tribes of South America were found to reckon by *fives*, which they called *hands*. In counting *six*, *seven*, and *eight*, they added to the word *hand* the names *one*, *two*, *three*, etc. Mungo Park found that the same system was practiced by the Yolofs and Foulahs of Africa, who designate ten by *two* hands, fifteen by *three* hands, etc. The quinary system seems also to have been formerly used in Persia; the word *pende*, which denotes *five*, having the same derivation as *péntcha*, which signifies a *hand*. It is even partially used in England among wholesale traders. In reckoning articles delivered at the warehouse, the person who takes charge of the *tale*, having traced a long horizontal line, continues to draw, alternately above and below it, a *warp*, or four vertical strokes, each set of which he crosses by an oblique score, and calls out *tally* as often as the number *five* is completed. This custom is a very general one in assemblies where votes are counted, and in similar circumstances elsewhere.

The *Senary* method, so far as we can learn, was never used by any tribe or nation; at least never arose spontaneously. It is said to have been adopted at one time in China by the order of a capricious tyrant, who, having conceived an astrological fancy for the number six, commanded its several combinations to be used in all concerns of business or learning throughout his vast empire.

The *Septenary* scale has not, so far as we can learn, been used anywhere. The number seven has been regarded as a kind of magic number, but nothing in nature suggested the method of counting by sevens. The division of the year into periods consisting of seven days each, a custom among nearly all nations, has given the number seven a wide distinction, and its frequent use in the Bible has caused it to be regarded as a sacred number, the basis of a celestial system of reckoning. The *Octary* scale, also, though it would possess many advantages, and has been recommended by scientific writers, has never made its appearance in any language. A *Nonary* scale has also never been used, and would be the most inconvenient of the smaller scales except the septenary.

The *Denary* scale is the system which has prevailed among all civilized nations, and has been incorporated into the very structure of their language. This universal method manifests the existence of some common principle of numbering, which was the practice, so familiar in the earlier periods of society, of reckoning by counting the fingers on both hands. The origin of the terms used in the more polished ancient languages is not easily traced, but in the roughness of savage dialects these names vary less from the primitive words. The Muysca Indians were accustomed to reckon as far as *ten*, which they called *quihicha* or a *foot*, referring, no doubt, to the number of toes on their bare feet; and beyond this number they used terms equivalent to *foot one*, *foot two*, etc., for *eleven*, *twelve*, etc. Another South American tribe called *ten*, *tunca*, and merely repeated the word to signify a *hundred*, or a *thousand*, thus: *tunca-tunca*, *tunca-tunca-tunca*. The Peruvian language was actually richer in the names of numerals than the Greek or Latin. The Romans went no higher than *mille*, a thousand, and the Greeks than *μυρια*, or ten thousand. But the Peruvians had the expressions, *huc*, one; *chunca*, ten; *pachac*, a hundred; *huaranca*, a thousand; and *hunu*, a million. It appears from an early document, that the Indian tribes of New England used the Denary scale, and had distinct words to express the numbers as far as a thousand. The Laplanders join the cardinal to the ordinal numbers; thus, for *eleven* they say *auft nubbe lokkai*, that is, *one to the second ten*. The origin of the numerals in our own dialect will be found treated at greater length in another place.

The mode of reckoning by *twelves* or *dozens*, may be supposed to have had its origin in the observation of the celestial phenomena, there being twelve months or lunations commonly reckoned in a solar year. The Romans likewise adopted the same number to mark the subdivisions of their unit of measure or of weight. The scale appears also in our subdivisions of weights and measures, as twelve ounces to a pound, twelve inches to a foot; and is still very generally employed in wholesale business, extending to the second and even to the third term of the progression. Thus, *twelve* dozen, or 144, make the *long hundred* of the northern nations, or the *gross* of traders; and twelve times this again, or 1728, make the *double* or *great gross*.

The scale of numeration by twenties has its foundation in nature, like the quinary and denary. In a rude state of society, before the discovery of other methods of numeration, men might avail themselves for this purpose, not merely of the fingers on the hands, but also of the toes on the naked feet; and such a practice would naturally lead to the formation of a vicenary scale of numeration. The languages of many tribes indicate this method, and many savage tribes do thus actually reckon. It is said of the inhabitants of the peninsula of Kamschatka, that "it is very amusing to see them attempt to reckon above ten; for having reckoned the fingers of both hands, they clasp them together, which signifies ten; then they begin at their toes and count twenty, after which they are quite confused and cry *matcha*, where shall I take more?" Among the Caribbees who constituted the native population of Barbadoes and other islands of the Caribbean sea, the numeration beyond five was carried on by means of the fingers and toes, and their numerical language became generally descriptive of their practical method of counting. The Abipones, an equestrian people of Paraguay, to express five show the fingers of one hand; to express ten, the fingers of both hands; "for twenty, their expression is pleasant, being allowed to show all the fingers of their hands and the toes of their feet."

Traces of reckoning by *scores* or *twenties*, are found in our own and other European idioms. The expression *threescore and ten* is familiar. The term *score* itself, which originally meant a *notch* or *incision* made on a tally to signify the successive completion of such a number, seems to indicate that such a mode of counting was most familiarly used by our ancestors. The vicenary scale seems to have prevailed very extensively among the Scandinavian nations, as is shown by the vestiges of it both among them and the languages partly derived from them. The French language has no term for the numbers in the second series of the denary scale above *soixante* or *sixty*. *Eighty* is expressed by *quatre-vingts, four twenties*, and *ninety* by *quatre-vingts-dix, four twenties and ten.* The people of Biscay and Armorica are said still to reckon by the powers of twenty, and, according to Humboldt, the same mode of numeration was employed by the Mexicans.

CHAPTER VI.

A DUODECIMAL SCALE.

AS already explained, any number may be made the basis of a system of numeration and notation. The decimal basis is a mere accident, and in some respects an unfortunate one, both for science and art. The duodecimal basis would have been greatly superior, giving greater simplicity to the science, and facilitating its various applications. In this chapter it will be explained how arithmetic might have been developed upon a duodecimal basis.

In order to make the matter clear, I call attention to two or three principles of numeration and notation. First, the bases of numeration and notation should be the same; that is, if we write numbers in a duodecimal system, we should also name numbers by a duodecimal system. Second, in naming numbers by any system, we give independent names up to the base, and then reckon by groups, using the simple names to number the groups. Bearing these principles in mind, we are ready to understand Numeration, Notation, and the Fundamental Rules in Duodecimal Arithmetic.

NUMERATION.—In naming numbers by the duodecimal system, we would first name the simple numbers from *one* to *eleven*, and then, adding one more unit, form a group, and name this group *twelve*. We would then, as in the decimal system, use these first names to number the groups. Naming numbers in this way, we would have the simple names, *one, two, three,* etc., up to *twelve*. Continuing from *twelve,* we would have *one and twelve, two and twelve, three and twelve,* etc., up to *twelve and twelve*, which we would call *two twelves*. Passing on from this we would have *two twelves and one*, *two twelves and two*, etc., to *three twelves*, and so on until we reach *twelve twelves*, when we would form a new group containing *twelve twelves*, and give this new group a new name, as *gross*, and then employ the first simple names again to number the *gross*. In this way we would continue grouping by twelves, and giving a new name to each group, as in the decimal scale by tens, as far as is necessary.

These names, in the duodecimal system, would naturally become abbreviated by use, as the corresponding names in the decimal system. Thus, as in the decimal system *ten* was changed to *teen*, we may suppose *twelve* to be changed to *teel*, and omitting the "and" as in the common system, we would count *one-teel*, *two-teel*, *thir-teel*, *four-teel*, *fif-teel*, *six-teel*, etc., up to *eleven-teel*. *Two-twelves* might be changed into *two-tel*, or *twen-tel*, corresponding to *two-ty* or *twenty*, and we would continue to count *twentel-one*, *twentel-two*, etc. *Three-twelves* might be contracted into *three-tel* or *thirtel*, corresponding to *three-ty* or *thirty* of the decimal system; *four-twelves* to *fourtel*, *five twelves* to *fiftel*, etc., up to a *gross*. Proceeding in the same manner, a collection of twelve gross would need a new name, and thus on to the higher groups of the scale.

In this manner, the names of numbers according to a duodecimal system could be easily applied. Were we actually forming such a system, the simplest method would be to introduce only a few new names for the smaller groups, and then take the names of the higher groups of the decimal system, with perhaps a slight modification in their orthography and pronunciation, to name the higher groups of the new scale. Thus, *million*, *billion*, etc., could be used to name the new groups without any confusion, as they do not indicate any definite number of units to the mind, but merely so many collections of smaller collections. Indeed, even the word *thousand*, with a modification of its orthography, say *thousun*, might be used to represent a collection of twelve groups, each containing a *gross*, without any confusion of ideas. Their etymological formation would not be an objection of any particular force, as no one in using them thinks of their primary signification. These terms are not suggested as the best, but as the simplest in making the transition from the old to the new system. It will also be noticed that our departure in the decimal scale from the principle of the system, by using the terms *eleven* and *twelve*, would facilitate the adoption of a duodecimal system.

To illustrate the subject more fully, let us adopt the names suggested, and apply them to the scale. Naming numbers according to the method explained, we would have the names as indicated in the following series:

one	oneteel	twentel-one	one gross and one
two	twoteel	twentel-two	one gross and two
three	thirteel	twentel-eight	two gross and five
four	fourteel	twentel-eleven	six gross and seven
five	fifteel	thirtel-one	ten gross and eight
six	sixteel	fortel-two	eleven gross and nine
seven	seventeel	fiftel-six	one thousun
eight	eighteel	sixtel-eight	one thousun and five
nine	nineteel	seventel-nine	one thousun four gross and seven
ten	tenteel	tentel-ten	two thousun seven gross and fortel-one
eleven	eleventeel	eleventel-eleven	
twelve	twentel	one gross	

NOTATION.—The writing of numbers by the duodecimal system would be an immediate outgrowth of the method of naming numbers in this system. As in the decimal system of notation, it would be necessary to employ a number of characters one less than the number of units in the base, besides the character for nothing. Since the group contains *twelve* units, the number of significant characters would be *eleven*—two more than in the decimal system. For these characters we should use the nine digits of the decimal system, and then introduce new characters for the numbers *ten* and *eleven*. To illustrate, we will represent *ten* by the character Φ and *eleven* by Π.

These characters, with the zero, would be combined to represent numbers in the duodecimal scale in the same manner as the nine digits represent numbers in the decimal scale. Thus, *twelve* would be represented by 10, signifying one of the groups containing *twelve;* 11 would represent *one and twelve,* or *oneteel;* 12 would represent *two and twelve,* or *twoteel;* 13 would represent *thirteel;* 14, *fourteel;* 15, *fifteel,* etc. Continuing thus, 20 would represent *two twelves,* or *twentel;* 21, *twentel-one;* 23, *twentel-three,* etc. The notation of numbers up to a *thousun* may be indicated as follows:—

one, 1	twelve, 10	thirtel, 30
two, 2	oneteel, 11	thirtel-two, 32
three, 3	twoteel, 12	thirtel-five, 35
etc., etc.	twentel, 20	thirtel-ten, 3Φ
nine, 9	twentel-one, 21	thirtel-eleven, 3Π
ten, Φ	twentel-ten, 2Φ	one gross, 100
eleven, Π	twentel-eleven, 2Π	one thousun, 1000

Extending the series as explained above, we should have the following notation table:—

TABLE.

Trillyuns.	Gross of Billyuns.	Twelves of Billyuns.	Billyuns.	Gross of Millyuns.	Twelves of Millyuns.	Millyuns.	Gross of Thousuns.	Twelves of Thousuns.	Thousuns.	Gross.	Twelves.	Units.
8	5	Π	4	6	Π	8	5	7	Φ	3	6	5

From the explanation given it is clearly seen that a system of duodecimal arithmetic might be easily developed, and readily learned and reduced to practice. Employing the names which I have indicated, or others similar to them, the change from the decimal to the duodecimal system would be much less difficult than has usually been supposed. It would be necessary to learn the method of naming and writing numbers, which we have seen is very simple, and a new addition and multiplication table, from which we could readily derive the elementary differences and quotients. The rest of the science would be readily acquired, as all of its methods and principles would remain unchanged. Indeed, so readily could the change be made, that in view of the great advantages of the system, one is almost ready to believe that the time will come when scientific men will turn their attention seriously to the matter and endeavor to effect the change.

FUNDAMENTAL OPERATIONS.—In order to show how readily the transition could be made, I will present the method of operation in the fundamental rules. We would proceed first to form an addition table containing the elementary sums, which, as in the decimal system, we would commit to memory. From this we could readily derive the elementary differences used in subtraction. Such a table is given on page 131.

By means of this table we can readily find the sum or difference of numbers expressed in the duodecimal system. To illustrate, required the sum of 487Π, 5Φ38, 63Π7, Φ856. The solution of this would be as follows: Adding the column of units, 6 units and 7 units are 11 units, and 8 units are 19 units, and Π units are 28 units, or 2 twelves and 8 units; writing the units, and carrying 2 to the column of twelves, we have 2 twelves and 5 twelves are 7 twelves, and Π twelves are 16 twelves, and 3 twelves are 19 twelves, and 7 twelves are 24 twelves, or 2 gross and 4 twelves; writing the twelves, and carrying 2 to the third column, we have 2 gross and 8 gross are Φ gross, and 3 gross are 11 gross, and Φ gross are 1Π gross, and 8 gross are 27 gross, or 2 thousuns and 7 gross; 2 thousuns and Φ thousuns are 10 thousuns, and 6 thousuns are 16 thousuns, and 5 thousuns are 1Π thousuns, and 4 thousuns are 23 thousuns; hence the amount is 23748.

OPERATION.
487Π
5Φ38
63Π7
Φ856
23748

To illustrate subtraction let it be required to find the differ-

ADDITION AND MULTIPLICATION TABLES IN THE DUODECIMAL SCALE.

ADDITION TABLE.

2+1=3	3+1=4	4+1=5	5+1=6	6+1=7	7+1=8	8+1=9	9+1=Φ	Φ+1=Π	Π+1=10	10+1=11
2+2=4	3+2=5	4+2=6	5+2=7	6+2=8	7+2=9	8+2=Φ	9+2=Π	Φ+2=10	Π+2=11	10+2=12
2+3=5	3+3=6	4+3=7	5+3=8	6+3=9	7+3=Φ	8+3=Π	9+3=10	Φ+3=11	Π+3=12	10+3=13
2+4=6	3+4=7	4+4=8	5+4=9	6+4=Φ	7+4=Π	8+4=10	9+4=11	Φ+4=12	Π+4=13	10+4=14
2+5=7	3+5=8	4+5=9	5+5=Φ	6+5=Π	7+5=10	8+5=11	9+5=12	Φ+5=13	Π+5=14	10+5=15
2+6=8	3+6=9	4+6=Φ	5+6=Π	6+6=10	7+6=11	8+6=12	9+6=13	Φ+6=14	Π+6=15	10+6=16
2+7=9	3+7=Φ	4+7=Π	5+7=10	6+7=11	7+7=12	8+7=13	9+7=14	Φ+7=15	Π+7=16	10+7=17
2+8=Φ	3+8=Π	4+8=10	5+8=11	6+8=12	7+8=13	8+8=14	9+8=15	Φ+8=16	Π+8=17	10+8=18
2+9=Π	3+9=10	4+9=11	5+9=12	6+9=13	7+9=14	8+9=15	9+9=16	Φ+9=17	Π+9=18	10+9=19
2+Φ=10	3+Φ=11	4+Φ=12	5+Φ=13	6+Φ=14	7+Φ=15	8+Φ=16	9+Φ=17	Φ+Φ=18	Π+Φ=19	10+Φ=1Φ
2+Π=11	3+Π=12	4+Π=13	5+Π=14	6+Π=15	7+Π=16	8+Π=17	9+Π=18	Φ+Π=19	Π+Π=1Φ	10+Π=1Π
2+10=12	3+10=13	4+10=14	5+10=15	6+10=16	7+10=17	8+10=18	9+10=19	Φ+10=1Φ	Π+10=1Π	10+10=20

MULTIPLICATION TABLE.

2×1=2	3×1=3	4×1=4	5×1=5	6×1=6	7×1=7	8×1=8	9×1=9	Φ×1=Φ	Π×1=Π	10×1=10
2×2=4	3×2=6	4×2=8	5×2=Φ	6×2=10	7×2=12	8×2=14	9×2=16	Φ×2=18	Π×2=1Φ	10×2=20
2×3=6	3×3=9	4×3=10	5×3=13	6×3=16	7×3=19	8×3=20	9×3=23	Φ×3=26	Π×3=29	10×3=30
2×4=8	3×4=10	4×4=14	5×4=18	6×4=20	7×4=24	8×4=28	9×4=30	Φ×4=34	Π×4=38	10×4=40
2×5=Φ	3×5=13	4×5=18	5×5=21	6×5=26	7×5=2Π	8×5=34	9×5=39	Φ×5=42	Π×5=47	10×5=50
2×6=10	3×6=16	4×6=20	5×6=26	6×6=30	7×6=36	8×6=40	9×6=46	Φ×6=50	Π×6=56	10×6=60
2×7=12	3×7=19	4×7=24	5×7=2Π	6×7=36	7×7=41	8×7=48	9×7=53	Φ×7=5Φ	Π×7=65	10×7=70
2×8=14	3×8=20	4×8=28	5×8=34	6×8=40	7×8=48	8×8=54	9×8=60	Φ×8=68	Π×8=74	10×8=80
2×9=16	3×9=23	4×9=30	5×9=39	6×9=46	7×9=53	8×9=60	9×9=69	Φ×9=76	Π×9=83	10×9=90
2×Φ=18	3×Φ=26	4×Φ=34	5×Φ=42	6×Φ=50	7×Φ=5Φ	8×Φ=68	9×Φ=76	Φ×Φ=84	Π×Φ=92	10×Φ=Φ0
2×Π=1Φ	3×Π=29	4×Π=38	5×Π=47	6×Π=56	7×Π=65	8×Π=74	9×Π=83	Φ×Π=92	Π×Π=Φ1	10×Π=Π0
2×10=20	3×10=30	4×10=40	5×10=50	6×10=60	7×10=70	8×10=80	9×10=90	Φ×10=Φ0	Π×10=Π0	10×10=100

ence between 6428 and 2564. We would solve this as follows: Subtracting 4 units from 8 units we have 4 units remaining; we cannot take 6 twelves from 2 twelves, so we add 10 twelves and have 12 twelves; 6 twelves from 12 twelves leaves 8 twelves; carrying 1 to 5 we have 6 gross; we cannot take 6 gross from 4 gross; adding 10 as before we have 6 gross from 14 gross leaves Φ gross; adding 1 thousun to 2 thousuns, we have 3 thousuns from 6 thousuns leaves 3 thousuns; hence the remainder is 3Φ84.

OPERATION.

```
6428
2564
----
3Φ84
```

In order to multiply and divide, we first form a multiplication table similar to that now used in the decimal system, and commit it to memory. This table need not extend beyond "twelve times," as in our present system there is no need of extending beyond "ten times." From this table of elementary products, we can readily derive the table of elementary quotients as we do in the decimal system. Such a table will be found on page 131.

It will be interesting to notice several peculiarities of this table, similar to those of the decimal system. As the column of "five times" ends alternately in 5 and 0, making it so easily learned by children, so the column of "six times" in the duodecimal table will end alternately in 6 and 0. In our present table the sum of the two terms of each product in the column of "nine times" equals nine, so in the duodecimal table, the sum of the two terms of each product in the column of "eleven times" equals eleven. We also notice that each product in the column of "twelve times" ends in 0, as does each product in the column of "ten times" of our present table.

By means of the multiplication table we can readily find the product or quotient of numbers expressed in the duodecimal scale. To illustrate multiplication, let it be required to find the product of 54Φ8 by 3Π7. We would solve this as follows: Using the first term of the multiplier, 7 times 8 are

48, 7 times Φ are 5Φ, and 4 are 62, 7 times 4 are 24 and 6 are 2Φ, 7 times 5 are 2Π and 2 are 31, making the first partial product 31Φ28; multiplying by Π we have Π times 8 are 74, Π times Φ are 92 and 7 are 99, Π times 4 are 38 and 9 are 45, Π times 5 are 47 and 4 are 4Π; 3 times 8 are 20, 3 times Φ are 26 and 2 are 28, 3 times 4 are 10 and 2 are 12, 3 times 5 are 13 and 1 are 14. Adding up the partial products, we have as the complete product, 1953768.

OPERATION.

```
    54Φ8
     3Π7
   ------
   31Φ28
  4Π594
 14280
 -------
 1953768
```

To illustrate division, let it be required to find the quotient of 1953768 divided by 3Π7. We would solve this as follows: We find that the divisor is contained in the first four terms of the dividend 5 times, and multiplying 3Π7 by 5 we have 179Π; subtracting this from the dividend we have a remainder, 174; bringing down the next figure of the dividend and proceeding as before, we have for the quotient 54Φ8.

OPERATION.

```
3Π7)1953768(54Φ8
    179Π
    ----
     1747
     13Φ4
     ----
      3636
      337Φ
      ----
       2788
       2788
       ----
```

The method of finding the square or cube root of a number expressed in the duodecimal scale is similar to that used in the decimal scale, as may be shown by an example. Thus, find the square root of Π5301. The greatest square in Π is 9; subtracting and bringing down a period, and dividing by 2 times 3 or 6, we find the second term of the root to be 4; completing the divisor and multiplying 64 by 4, we have 214; subtracting and bringing down, we have 3Π01, and dividing by 2 times 34, or 68, we have 7 for the last figure of the root; completing the divisor and multiplying it by 7, we have 3Π01, which leaves no remainder.

OPERATION.

```
      Π·53·01(347
      9
     ----
 64)253
    214
    -----
687)3Π01
    3Π01
    ----
```

The above tables and calculations seem awkward to one who is familiar with the decimal system; but it should be remembered that a beginner would learn the addition and multiplication tables and the calculations based on them, just as readily as he now learns them in the decimal system. The practical value of such a system, in addition to what has already been said, may be seen in the calculation of interest, the rules for which would be greatly simplified on account of the relation of the number of months in a year (12) to the base, and also of the relation of the rate to the same, which would be some 8% or 9%; that is, 8 or 9 per gross. I hope to be able in a few years to publish a small work in which the whole science of arithmetic shall be developed on the duodecimal basis.

CHAPTER VII.

GREEK ARITHMETIC.

GREEK Arithmetic, like that of all other nations of antiquity, began in the representation of numbers by strokes or straight lines. This system, in the progress of thought and civilization, was finally discarded, and the letters of the alphabet taken as the symbols of numbers. After adopting the letters of their alphabet, the Greeks seem to have had no less than three distinct methods of notation. They used the letters in their natural order, to signify the smaller ordinal numbers. In this way the books of Homer's *Iliad* and *Odyssey* are usually marked. They employed also the first letters of the words for numerals as abbreviated symbols, making use of an ingenious device for augmenting the powers of these symbols; thus, a letter enclosed by a line on each side and another drawn over the top, as ⨅, was made to signify five thousand times its original value.

A more complete method consisted in the distribution of the twenty-four letters of their alphabet into three classes, corresponding to units, tens, and hundreds, adding another character to each class to complete the symbols for all of the nine digits. This latter method was the one in common use, and that which was made the basis of their arithmetic. The units from one to nine inclusive, were denoted by the letters α, β, γ, δ, ε, ς, ζ, η, θ; the tens by ι, κ, λ, μ, ν, ξ, ο, π, ϟ; and the hundreds by ρ, σ, τ, υ, φ, χ, ψ, ω, ϡ. Thousands were represented by the first series with the *iota*, or *dash* subscribed, thus: α̦, β̦, γ̦, δ̦ etc. With these characters they could readily express any number under

10,000, or a myriad. Thus, 991 was expressed by ϡϟα; 7382, by ζτπβ; 6420, by ͵ςυκ; 4001, by ͵δα.

It will be noticed that neither the order nor the number of characters was considered in expressing numbers. The value of the expression was the same in whatever order the letters were placed; though as regularity tended towards simplicity, they generally wrote the characters according to value, from left to right.

Myriads, or ten thousands, were denoted by the letter M, a letter representing the number of myriads indicated being written above it. Thus, $\overset{\alpha}{M}$ denoted 10,000; $\overset{\beta}{M}$, 20,000; $\overset{\gamma}{M}$, 30,000, etc. Thus, also, $\overset{\lambda\zeta}{M}$ denoted 370,000; $\overset{\delta\tau o\beta}{M}$, 43720000; and in general, the letter M placed beneath any number had the same effect as our annexing four ciphers.

This is the notation employed by Eutocius in his commentaries on Archimedes, but it is evidently inconvenient in calculation. Diophantus and Pappus expressed the myriad more simply by the two letters Mυ placed after the number, and afterwards by merely writing a point after it. This enabled them to express 100,000,000, which was the greatest extent of the ordinary Greek arithmetic.

This system had been extended by Archimedes and Apollonius, for the purpose of astronomical and other scientific calculations. Archimedes, in order to express the number of grains of sand that might be contained in a sphere that had for its diameter the distance of the fixed stars from the earth, found it necessary to represent a number which, with our notation, would require sixty-four places of figures; and in order to do this, he assumed the square myriad, or 100,000,000, as a new unit, and the numbers formed with these new units he called numbers of the second order; and thus he was enabled to express any number which in our notation requires sixteen figures. Assuming again $100,000,000^2$ as a new unit, he could represent any number that requires in our scale twenty-four figures, and so on; so that by means of his numbers of the eighth order, he could express the number in question, which requires sixty-four figures in our scale.

By this system all numbers were separated into periods or orders of eight figures. This was afterwards considerably improved by Apollonius, who, instead of periods of eight places, which were called by Archimedes *octates*, reduced numbers to periods of four places; the first of which, on the left, were units, the second period myriads, the third double myriads or numbers of the second order, and so on indefinitely. In this manner Apollonius was able to write any number that can be expressed by our system of numeration; as for example, if he had wished to represent the circumference of a circle whose diameter was a myriad of the ninth order, he would have written it thus:

γ.αυιε.	θσξε.	χφπθ.	ϛϡλβ.	χωμϛ.	βχμγ.	χωλβ.	ϛϡν.	βωκδ.
3.1415	9265	3589	7932	3846	2643	3832	7950	2824

The learned astronomer Ptolemy modified this system in its descending range by applying it to the sexagesimal subdivisions of the lines inscribed in a circle. He likewise advanced an important step, by employing a small or accentuated *o* to supply the place of any number wanting in the order of progression.

The Greek method of expressing fractions was also peculiar. An accent set on the right of a number, made of that number the denominator of a fraction whose numerator was a unit, thus, $\gamma' = \frac{1}{3}$, $\delta' = \frac{1}{4}$, $\xi\delta' = \frac{1}{64}$, $\rho\kappa\alpha' = \frac{1}{121}$, etc. When the numerator is not unity, the denominator is placed as we set our exponents. Thus, $\iota'\epsilon^{\xi\delta}$ represented 15^{64}, or $\frac{15}{64}$, and $\zeta^{\rho\kappa\alpha}$ represented 7^{121}, or $\frac{7}{121}$. The fraction $\frac{1}{2}$ had a particular character, as C, <, C′, or K. The notation of the Greeks was not adapted to the descending scale, and consequently they had no decimals.

The notation of the Greeks, though much inferior to that of the present day, was formed upon a regular and scientific basis, and could be employed with considerable convenience as an instrument of calculation. We will present two or three examples taken from Barlow's *Theory of Numbers*, from which some of the previous facts are gathered.

Addition.—The following example in addition is from Eutocius, Theorem 4, of the Measure of the Circle.

ωμζ.γϡκα	847 3921
ξ . ηυ	60 8400
ϡη .βτκα	908 2321

The method, it will be seen, is similar to compound addition, but is simpler on account of the constant ratio of ten between any character and the succeeding one.

Subtraction.—The following example in subtraction is from Eutocius, Theorem 3, on the Measure of the Circle.

θ.γχλϛ	93636
β.γυ θ	23409
ζ. σκζ	70227

The method is simple, proceeding from right to left as in our subtraction, which seems so obviously advantageous and simple that one can hardly conceive why the Greeks should ever proceed in the contrary way, although there are many instances which make it evident that they did, both in addition and subtraction, work from left to right.

Multiplication.—In multiplication they most commonly proceeded in their operations from left to right, as we do in multiplication in algebra, and their successive products were placed without much apparent order; but as each of their characters retained always its own proper value, in whatever order they stood, the only inconvenience of this was, that it rendered the addition of them a little more troublesome.

ρνγ	$1''5'3$
ρνγ	$1''5'3$
α.ετ	$1^{m}5'''3''$
εβφρ	$5'''2'''5''1''5'$
τρνθ	$3''1''5'9^{\circ}$
β.γυθ	$2^{m}3'''4''$ 9°

The following example is from Eutocius. As it is difficult to remember the value of all the Greek characters, we will indicate the operation by writing 1°, 2°, 3°, etc., for the series of units

1′, 2′, 3′, etc., for the series of tens; 1″, 2″, etc., for the hundreds, etc., and denote the myriads by writing *m* as an exponent.

Division.—The division of the Greeks was still more intricate than their multiplication, for which reason it seems they generally preferred the sexagesimal division, and no example is left at length by any of those writers except in the latter form; but these are sufficient to throw some light on the process they followed in the division of common numbers, and Delambre has accordingly supposed the following example:

τλβ.γτκθ(	αωϛγ	$332^{m}\,3'''3''2'9^{\circ}($	$1'''8''2'3^{\circ}$
ρπβ.γ	αωκγ	182 3	$1'''8''2'3^{\circ}$
ρν.τκθ		150 0 3 2 9	
ρμε.ηυ		145 8 4	
δ.αϡκθ		4 1 9 2 9	
γ.ϛυξ		3 6 4 6	
ευξθ		5 4 6 9	
ευξθ		5 4 6 9	

This example will be found, on a slight inspection, to resemble our compound division, or that sort of division that we must necessarily employ, if we were to divide feet, inches and parts by similar denominations, which, together with the number of different characters that they made use of, must have rendered this rule extremely laborious; and that for the extraction of the square root was, of course, equally difficult, though the principle was the same as ours, except in the difference of the notation. It appears, however, that they frequently, instead of making use of the rule, found the root by successive trials, and then squared it in order to prove the truth of their assumption.

This beautiful system was vastly superior in simplicity and practical utility to that transmitted to and retained by the Romans, and by them bequeathed to the nations of modern Europe. It was, at least when it had reached its highest development through the genius of Archimedes and Apollonius, quite well fitted for an instrument of calculation; and though somewhat cumbrous in its structure, was capable of performing operations of very considerable difficulty and magnitude.

It will be seen, however, that though much more refined and pliant than that of the Romans, the notation of the Greeks is very much inferior to the common or Hindoo method; and one cannot help wondering that so ingenious and philosophical a people failed to conceive the simple idea of *place value*, and construct a system of notation upon it. This seems all the more astonishing when we remember that Archimedes invented a system of *octates*, or system of eights, which was subsequently improved by Apollonius, by making the periods consist of only four places, and dividing all numbers into orders of myriads. In this form, as Barlow remarks, it seems most astonishing that he did not perceive the advantage of making the periods to consist of a less number of characters; for, having given a local character to his periods of four, it was only necessary to have done the same for the single digits, in order to have arrived at the system in present use. And this is the more singular, as the use of the cipher was not unknown to the Greeks, being always employed in their sexagesimal operations where it was necessary; and consequently the step between this improved form of their notation and that of the present system was extremely small, although the advantages of the latter when compared with the former are incalculably great. It seems to have been the lot of the metaphysical mind of the Hindoos to make this "brilliant invention of the decimal scale," one of the greatest improvements in the whole circle of the sciences, and to which we are indebted for all the remarkable advances in modern analysis.

CHAPTER VIII.

ROMAN ARITHMETIC.

THE arithmetic of the Romans was quite inferior to that of the Greeks, a necessary consequence of the inferiority of the method of notation adopted. The method of notation, though usually ascribed to the Romans, was probably invented by the Greeks, and communicated by them to the Romans, who in turn transmitted it to their successors in modern Europe. It no doubt originated in the use of simple strokes, variously combined, to represent numbers. Subsequently it was found convenient to represent numbers by the letters of the alphabet, and the numerical strokes were finally displaced by such alphabetic characters as most nearly resembled them.

The origin of the Roman characters is not certainly known; but the theory, as given by Leslie, and by many regarded as correct, is interesting and plausible. It is certain that the first numerical characters consisted simply of strokes or straight lines. This was the method primarily used by nearly every nation of antiquity, and was the beginning of a philosophical and universal system alike intelligible to all nations. Such characters are still preserved in the Roman notation with very little change, and were probably adopted before the importation of the alphabet itself, by the Grecian colonies that settled Italy and founded the Latin commonwealth. Assuming, then, a perpendicular line | to signify *one*, two such lines | | to signify *two*, three lines | | | to signify *three*, and so on up to *ten*, and we have the first series of the numerical scale. They might then be supposed to throw a dash across the last stroke or unit, to mark the completion of the series; and thus, a cross, ×, would come to signify *ten*. The continued repetition of this mark would denote *twenty*, *thirty*, etc., until they reached a hundred, or *ten* tens, which completes the second series, and might be denoted by adding another dash to the mark for ten, or by merely connecting three strokes, thus ⊏. The repetition of this symbol would, in like manner, indicate the successive hundreds, the tenth of which would be marked by the addition of another stroke, so that four combined strokes, M, would express a thousand.

Such were probably the symbols originally employed in the Roman notation; in process of time it would be perceived that the inconvenience in writing, arising from so many repetitions of the same character, might be avoided by adopting symbols for the intermediate numbers; and it was seen that these might be furnished by the division of the symbols already in use. Thus, having parted in the middle the two strokes, ×, either the under half, ∧, or the upper half, ∨, was employed to signify *five*, or the half of *ten*. Next, for *fifty*, the half of a hundred, the symbol ⊏ was divided into two equal parts, ┌ and └, either of which represented *fifty*. Again, the symbol for *thousand* having come to assume a rounded shape, thus ↀ, or thus CIↃ, the half of this, either CI, or IↃ, was taken to represent the half of one thousand or *five hundred*. The symbol ⊏, to represent a hundred, would, in process of time, being frequently made, have its corners rounded and attain the form C. Lastly, noticing that these characters closely resemble some of the letters of the alphabet, it was agreed to employ those letters as the symbols of the numbers mentioned.

The notation of numbers by combined strokes, was evidently founded in nature, and may be regarded as the beginning of a philosophical language of arithmetic. That this was the foundation of the Roman system is confirmed from the analogous practice of other nations. It is quite clear that the

Egyptians and Chinese must have followed the same method. The inscriptions on the ancient obelisks present a few numerals which are easily distinguished. The substitution of capital letters for the combined strokes which they chanced most to resemble, though it gave uniformity to the system of notation, prevented any farther improvements of the system. The only simplification which the Romans appear to have introduced, was to diminish the repetition of letters by reckoning in some cases backwards, as in IV, which was originally represented by four strokes, and IX, which was probably at first written VIIII.

Their method of representing large numbers was a little different from that now used, as may be seen by the following examples:

D or IↃ	M or CIↃ	IↃↃ	CCIↃↃ	IↃↃↃ	CCCIↃↃↃ
500	1000	5000	10,000	50,000	100,000.

In illustration, it is interesting to notice that Cicero in his fifth oration against Verres expresses 3600 by CIↃ CIↃ CIↃ IↃC. The Romans often contracted or modified the forms of their numerals, especially in carving inscriptions upon stones, in which case the abbreviated letters were called *lapidary characters.*

The marks for any number could also be augmented in power *one thousand* times, either by enclosing them with two hooks or C's, or by drawing a line over them. Thus, CXↃ, or $\overline{\text{X}}$ denoted 10,000, and $\overline{\text{CLVIM}}$ given by Pliny, means 156,000,000. Sometimes a letter was placed over another to indicate their product; thus, $\overset{\text{D}}{\text{M}}$ would express 500,000. The multiplier was also sometimes written like an exponent, thus III^{c} was used to express *three hundred.* In expressing very large numbers, points were sometimes interposed: thus, Pliny writes XVI. XX.DCCCXXIX for 1,620,829. It may be remarked that if this practice had become more general it would probably have effected a material improvement of the system.

In the latter ages of the Roman Empire, the small letters of the alphabet seem to have been used in imitation of the numeral system of the Greeks. The letters a, b, c, d, e, f, g, h, and i represented the nine digits 1, 2, 3, 4, 5, 6, 7, 8, and 9; the next series k, l, m, n, o, p, q, r, and s expressed 10, 20, 30, 40, 50, 60, 70, 80, and 90; and the remaining letters t, u, x, y, and z denoted 100, 200, 300, 400, and 500. To represent the rest of the hundreds it was necessary to employ capitals or other characters, and 600, 700, 800, and 900 were represented by I, V, hi and hu. But this mode of notation never obtained any degree of currency, being mostly confined to those foreign adventurers from Greece, Egypt or Chaldea, who, pretending to skill in judicial astrology, were enabled to prey on the credulity of the wealthy Romans.

In modern Europe the Roman numerals were supplied by Saxon characters. Thus, in the accounts of the Scottish Exchequer for the year 1331, the sum of £6896 5s. 5d. stated as paid to the King of England is thus marked:

ꝯ c xx
vj. viij. iiij. xvj. ŧj. v. s̃. v. đ.

The Roman system, as now used, employs seven characters, of which I represents one, V five, X ten, L fifty, C one hundred, D five hundred, M one thousand. To express other numbers these characters are combined according to the following principles:—

1. Every time a letter is repeated its value is repeated.

2. When a letter is placed after one of greater value, the sum of their values is the number expressed.

3. When a letter is placed before one of a greater value, the difference of their values is the number expressed.

4. When a letter stands between two letters of a greater value, it is combined with the one following it.

5. A letter is placed before one of its own order only, or the *unit* of the next higher order.

6. A dash over a letter increases its value a thousand fold.

In accordance with the fifth principle it would be incorrect to write VC for *ninety-five*, or IC for *ninety-nine*. It is also to be noticed that the letter V is never used before a letter of greater value, since the only case in which it could be thus used according to the fifth principle is before X, giving VX for *five*, which is more concisely expressed by V itself.

In expressing numbers by the Roman method we always *write the different orders of units successively, beginning with the higher orders*. Thus, in expressing *four hundred and ninety-nine*, we would not write ID, though this, by principle second, would be the difference of *one* and *five hundred*, but we first write CCCC for *four hundred*, then XC for *ninety*, and then IX for *nine*, giving CCCCXCIX.

It may be interesting to notice, however, that though the Roman method was not employed in numerical calculations, it might have been so employed by slightly modifying the usual mode of notation. Thus, by not using the third principle, but writing IIII for IV, and VIIII for IX, or by using some mark to show that the letters written according to that principle are taken together, as XXIV, we can perform the four fundamental operations without much inconvenience. To illustrate, we give a problem in multiplication, with its explanation.

OPERATION.

```
          LXVIII
          XXXVII
     -----------
     CCCLLXXXLVI
     DCLXXX
   DCLXXX
 DCLXXX
 ---------------
 MMD         XVI
```

Explanation.—VIII multiplied by VII equals LVI, X multiplied by VII equals LXX, L multiplied by VII equals CCCL; III multiplied by X equals XXX, X multiplied by X equals C, L multiplied by X equals DCL; multiplying by X a second and third time, and taking the sum of the four partial products, we have MMDXVI, or two thousand five hundred and sixteen. This result may be obtained by multiplying by VII and XXX; or by II, V, X, and XX, etc. The multiplicand also may be variously separated in the multiplication.

It is clear, however, that this operation would be very complicated with large numbers, so much so, indeed, as to be unfitted for general use, and it is believed that it was not used in performing numerical calculations. These calculations were performed by means of counters, or other palpable emblems. The instrument generally used was called the *Abacus*. Leslie says that "the system of characters among the Romans was so complex and unmanageable as to reduce them to the necessity in all cases of employing the *Abacus*."

The Abacus appears to have continued in use among the people of Europe until quite a recent period. The counters or pebbles were, from a corruption of the word *algorithm*, called in England *augrim*, or *awgrym*, *stones*. Thus, in Chaucer's description of the chamber of Clerk Nicholas, he says:

> "His almageste and bokes grete and smale,
> His astrelabre, longing for his art,
> His augrim stones layen faire apart
> On shelves couched at his beddes head."

Indeed, the modern method of arithmetic was not known in England until about the middle of the sixteenth century; and the common people, imitating the clerks of former times, were still accustomed to reckon by the help of the *awgrym stones*. Thus, in Shakespeare's comedy of the *Winter's Tale*, written at the beginning of the seventeenth century, a clown, staggered at a very simple multiplication, exclaims that he must try it with counters.

CLO. Let me see; Every 'leven wether — tods; every tod yields — pound and odd shilling; fifteen hundred shorn,—What comes the wool to? . . . I cannot do't without counters.

The Roman method is now chiefly used to denote the volumes, chapters, sections and lessons of books, the pages of prefaces and introductions, to express dates, to mark the hours on clock and watch faces, and in other places for the sake of prominence and distinction.

CHAPTER IX.

PALPABLE ARITHMETIC.

THE earliest methods of representing numbers in arithmetical calculation were by means of counters and other palpable emblems. The objects most generally used among all primitive nations were little stones or pebbles, from which we derive our word *calculation*. Beginning with pebbles or some such simple objects, as they advanced in civilization these were found to be insufficient for their purposes, and they invented instruments to represent numbers, by means of which they were enabled to calculate with great rapidity and correctness. The Japanese and Chinese at the present day, with their arithmetical instruments, can add, subtract, multiply and divide as rapidly and correctly as we can with the Arabic system of notation. So extensively was this method used by the early nations before the method of calculating by figures was adopted, that Leslie, in his treatise on arithmetic, gives it a distinct and detailed explanation under the head of *Palpable Arithmetic*. The subject is so full of interest, both for its own ingenuity and its relation to our present system, that I think it proper to devote a chapter to it, and finding a clearer statement of it in Leslie and Peacock than I could hope to give myself, I have transcribed their description, sometimes word for word.

The early Egyptians performed their computations mainly by the help of pebbles, and so did the early Greeks and Romans. In the schools of ancient Greece, the boys acquired the elements of knowledge by working on the ABAX, a smooth board with narrow rim, so named evidently from the combination of the first three letters of their alphabet, and resembling the tablet on which children were formerly accustomed to begin to learn the art of reading. Pupils were taught to calculate by forming progressive rows of counters, which consisted of round bits of bone or ivory, or even silver coins, according to the wealth or fancy of the individual. The same board, strewed with fine green sand, a color soft and agreeable to the eye, served equally for teaching the rudiments of writing and the principles of geometry.

The ancient writers make frequent allusions to these calculating boards. Solon, the great Athenian statesman, used to compare the passive ministers of kings to the counters or pebbles of arithmeticians which, according to the place they hold, are sometimes most important, and sometimes utterly insignificant. The Grecian orators, in speaking of balanced accounts, picture the settlements by saying that the pebbles were cleared away and none left. It thus appears that the ancients, in keeping their accounts, did not arrange the debits and credits separately, but set down pebbles for the former, and took up pebbles for the latter. As soon as the board became cleared, the opposite claims were exactly balanced. It may be observed that the common phrase *to clear one's scores or accounts*, meaning to settle or adjust them, still preserved in the popular language of Europe, was suggested by the same practice of reckoning with counters, which prevailed, indeed, until a comparatively late period.

The Romans borrowed their *Abacus* from the Greeks, and seem never to have aspired higher in the pursuit of numerical science. To each pebble or counter required for the board they gave the name of *calculus*, meaning a small white stone, and applied the verb *calculare* to express the operation of combining or separating such pebbles or counters. The use of the *Abacus*, called also the *Mensa Pythagorica*, formed an essential part of the education of every noble youth. A small box or coffer, called a *Loculus*, having compartments for holding the *calculi*, or counters, was considered as a necessary appendage. Instead of carrying a slate and satchel to school, the Roman boy was accustomed to trudge to school loaded with those ruder implements,—his arithmetical board and his box of counters.

In the progress of luxury and refinement, dice made of ivory, called *tali*, were used instead of pebbles, and small silver coins came to supply the place of counters. Under the Emperors, every patrician living in a spacious mansion and indulging in all the pomp and splendor of Eastern princes, generally entertained, for various functions, a numerous train of foreign slaves or freedmen in his palace. Of these, the *librarius*, or *miniculator*, was employed in teaching the children their letters, the *notarius* registered expenses, the *rationarius* adjusted and settled accounts, and the *tabularius* or *calculator*, working with his counters and board, performed what computations might be required.

To facilitate the working by counters, the construction of the *Abacus* was afterwards improved. Instead of the perpendicular lines, or bars, the board had its surface divided by sets of parallel grooves, by stretched wires, or even by successive rows of holes. It was easy to move small counters in the grooves, to slide perforated beads along the wires, or to stick large knobs or round-headed nails in the different holes. To diminish the number of marks required, every column was surmounted by a shorter one, wherein each counter had the same value as five of the ordinary kind. The *Abacus*, instead of wood, was often, for the sake of convenience and durability, made of metal, frequently brass, and sometimes silver. Two varieties of this instrument seem to have been used by the Romans. Both of them are delineated from antique monuments—the first kind by Ursinus, and the second by Marcus Velserus. In the former, the numbers are represented by flattish perforated beads, ranged on parallel wires; and in the latter, they are signified by small round counters, moving in parallel grooves. These instruments contain each seven capital divisions, expressing in regular order *units*, *tens*, *hundreds*, *thousands*, *ten thousands*, *hundred thousands*, and *millions*, and as many shorter divisions, of five times the relative value of the larger ones. With *four* beads on each of the long grooves or wires, and *one* on each corresponding short one, it is evident that any number could be expressed up to *ten* millions. The Roman *Abacus* also contained grooves to mark ounces, half-ounces, quarter-ounces, and thirds of an ounce.

The Romans likewise applied the same word *Abacus* to an article of furniture resembling in shape the arithmetical board, but often highly ornamented, which was destined for the amusement of the opulent. It was used in a game apparently similar to that of chess, in which the infamous and abandoned Nero took particular delight, driving over the surface of the *Abacus* with a beautiful ivory *quadriga* or chariot.

The Chinese have, from the remotest ages, used in all their computations, an instrument similar in shape and construction to the Roman *Abacus*, but more complete and uniform. It is admirably adapted to the decimal system of weights, measures, and coins, which prevails throughout the empire. The whole range includes ten bars, and the calculator may begin at any one and reckon upwards or downwards with equal facility, treating fractions exactly like integers—an advantage of the utmost consequence in practice. Accordingly these arithmetical machines, of various sizes, have been adopted by all ranks, from the man of letters to the humblest shopkeeper, and are constantly used in all the bazaars and booths of Canton and other cities, being handled, it is said, by the native traders with a rapidity and address quite astonishing.

Among the various nations which regained their independence by the fall of the Roman Empire, it was found convenient in all transactions where money was concerned, to follow the procedure of the *Abacus*, in representing numbers by counters placed in parallel rows. During the Middle Ages, it became the usual practice over Europe for merchants, auditors of accounts, or judges appointed to decide in matters of revenue, to appear on a covered bank or bench, so called from an old Saxon or Franconian word signifying a *seat*. The term *scaccarium*, a Latinized Oriental word, from which was derived the French and then the English name for the *Exchequer*, anciently indicated merely a *chess-board*, being formed from *scaccum*, one of the pieces in that game.

The Court of Exchequer, which takes cognizance of all questions of revenue, was introduced into England by the Norman Conquest. Fitz-Nigel, in a dialogue on the subject, written about the middle of the twelfth century, says that the *scaccarium* was a quadrangular table about ten feet long and five feet broad, with a ledge or border about four inches high, to prevent anything from rolling over, and was surrounded on all sides by seats for the judges, the tellers, and other officers. It was covered every year, after the term of Easter, with fresh black cloth, divided by perpendicular white lines or distinctures, at intervals of about a foot or a palm, and again parted by similar transverse lines. In reckoning accounts, they proceeded according to the rules of arithmetic, using small coins for counters. The lowest bar exhibited *pence*, the one above it *shillings*, the next *pounds*, and the higher bars denoted successively *tens*, *twenties*, *hundreds*, *thousands*, and *ten thousands* of pounds; though, in those early times of penury and severe economy, it very seldom happened that so large a sum as the last ever came to be reckoned. The teller sat about the middle of the table; on his right hand, *eleven* pennies were heaped on the first bar, and *nineteen* shillings on the second, while a quantity of pounds was collected opposite to him, on the third bar. For the sake of expedition he might employ a different mark to represent half the value of any bar, a silver penny for ten shillings, and a gold penny for ten pounds.

In early times, a *checkered board*, the emblem of calculation, was hung out, to indicate an office for changing money. It was afterwards adopted as the sign of an inn or *hostelry*, where victuals were sold, or strangers lodged and entertained. It is said that traces of this ancient practice may be found even at the present day.

The use of the smaller *Abacus* in assisting numerical computation was not unknown during the Middle Ages. In England, however, it appears to have scarcely entered into actual practice, being mostly confined to those few individuals who, in such a benighted period, passed for men of science and learning. The calculator was styled, in correct Latin, *abacista;* but in Italian, *abbachista*, or *abbachiere*. The Arabians, having adopted an improved species of numeration, to which they gave the barbarous name of *algarismus* or *algorithmus*, from their definite article *al*, and the Greek word for *number*, this compound term was adopted by the Christians of the West, in admiration of their superior skill, to signify calculation in general, long before the peculiar method of performing it had become known and practiced among them. The term *algarism* was converted in English into *augrim* or *awgrym*, and applied even to the pebbles or counters used in ordinary calculation. The same word, *algorithm*, is now applied by mathematicians to express any peculiar sort of notation.

The *Abacus* had been adopted merely as an instrument for facilitating the process of computation. It became necessary, however, to adopt some simpler and more convenient method of expressing numbers. A very ancient practice consisted in employing the various articulations and dispositions of the fingers and the hands, to denote the numerical series. On this narrow basis, the Romans framed a system of considerable extent. By the inflexion of the various fingers of the left hand, they proceeded as far as *ten*, and by combining these with some other given inflexions, as changes in the position of the thumb, they could advance to a hundred; and using the right hand in a similar manner, they proceeded as far as a *thousand* and *ten thousand.* This is as far as the system appears to have been carried by the ancients; but the venerable Bede, by referring these signs to the various parts of the body, as the head, the throat, the side of the chest, the stomach, the waist, the thigh, etc., has shown how they could be again multiplied a hundred times, and raised to the extent of a million. In this numerical play, the Romans especially had acquired great dexterity. Many allusions to the practice are made by their poets and orators, and without some knowledge of the principle adopted, many passages of the classics would lose their whole force.

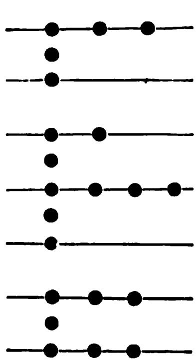
A species of digital arithmetic seems to have existed among nearly all the Eastern nations. The Chinese have a system of indigitation by which they can express on one hand all numbers less than 100,000 The thumb nail of the right hand touches each joint of the little finger, passing first up the external side, then down the middle, and afterwards up the other side of it, in order to express the nine digits; the tens are denoted in the same way on the second finger; the hundreds on the third; the thousands on the fourth; the tens of thousands on the thumb. It would be only necessary to proceed to the right hand in order to be able to extend this system of numeration much further than could be required for any ordinary purposes. The Bengalese count as far as 15 by touching in succession the joints of the fingers; and merchants in concluding bargains, the particulars of which they wish to conceal from the by standers, put their hands beneath a cloth and signify the prices they offer or take by the contact of the fingers. The same custom is prevalent also in Barbary and Arabia, where they conceal their hands beneath the folds of their cloaks, and possess methods which are probably peculiar and national, of conveying the expression of numbers to each other.

Juvenal states it as a peculiar felicity of Nestor that he counted the years of his age on his right hand. The image of Janus was represented, according to Pliny, with the fingers so placed as to represent 365, the number of days in the year. Some authors have supposed that Solomon in the passage, "Length of days is in her right hand, and in her left hand riches and honor," referred to this practice. The common phrases, *ad digitos redire, in digitos mittere*, have the same meaning as *computare*, and distinctly refer to digital numeration; and the phrase *micare digitis*, of frequent occurrence, alludes to a game extremely popular among the Romans, and which was probably the same as the *morra* of modern Italy. This noisy game is played by two persons, who stretch out a number of their fingers at the same moment, and instantly call out a number; and he is the winner who expresses the sum of the number of fingers thrown out. The same game is found amongst the Sicilians, Spaniards, Moors, and Persians, and under the name *tsoimoi*, is practiced also in China.

These signs were merely fugitive, and it became necessary to adopt other marks of a permanent nature for the purpose of recording numbers. But of all the contrivances adopted with this view, the rudest undoubtedly is the method of registering by *tallies*, introduced into England along with the Court of Exchequer, as another badge of the Norman Conquest. These consist of straight, well-seasoned sticks of hazel or willow, so called from the French verb *tailler, to cut*, because they are squared at each end. The sum of money was marked on the side with notches, by the cutter of tallies, and likewise inscribed on both sides in Roman characters, by the writer of the tallies. The smallest notch signified a penny, a larger one a shilling, and one still larger a pound; but other notches, increasing successively in breadth, were made to denote *ten*, a ***hundred***, and a *thousand*. The stick was then cleft through the middle by the deputy-chamberlains, with a knife and mallet, the one portion being called a *tally*, or sometimes the *scachia, stipes*, or *kancia*, and the other portion named the *counter-tally* or *folium.*

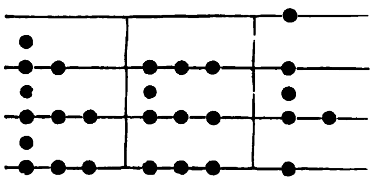
This strange custom might seem the practice of untutored Indians, and can be compared only to the rude simplicity of the ancient Romans, who kept their diary by means of *lapilli* or small pebbles, casting a white pebble into the urn on fortunate days, and dropping a black one when matters looked unprosperous; and who sent, at the close of each year, the Prætor Maximus, with great solemnity, to drive a nail in the door of the right side of the temple of Jupiter, next to that of Minerva, the patron of learning and inventor of numbers.

The use of counters was general throughout Europe as late as the end of the 15th century: about that period they were no longer used in Italy and Spain, where the early introduction of the Arabic figures and the number of treatises on the use of these figures had rendered them unnecessary. Recorde, in his *Ground of Arts*, prefaces his second dialogue, entitled "The Accounting by Counters," by observing, "Now that you have learned Arithmetic with the pen, you shall see the same art in counters, which feat doth not onely serve for them that cannot write and read, but also for them that can do both, but have not at the same time their pen or tables with them."



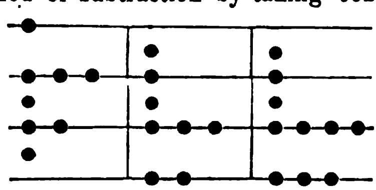
We shall now proceed to give some account of the method of performing operations by this *palpable* or *calcular arithmetic.* They commenced by drawing seven lines with a piece of chalk, on a table, board, or slate, or by a pen on paper, as in the margin; the counters, which were usually of brass, on the lowest line represented units, on the next tens, and so on as far as millions on the uppermost line; a counter placed between two lines represented five counters on the line next below it; thus, the number represented in the margin is 3629638, and the number of lines may evidently be increased so as to represent any number.

To add two numbers, such as 788 and 383, we divide the lines as in the margin, so as to form three columns, writing the first number in the first column, numbering from the left, the second in the second, and the result in the third column. The sum of the counters on the lowest line in the first two columns is 6; we therefore place one on that line in the third column, and carry one to the space above which, added to the one already there, makes one on the second line; adding this counter to the six already there, we have 7, and therefore place 2 on the line and carry one to the space above; adding the counters on that space, we find there are 3, hence we leave one in the space and carry one to the next line, in which the sum of the counters is six; we leave one on the line and carry one to the space above, and adding to the counter already there we have two counters, hence we leave no counter there, but place one on the fourth line; the sum thus obtained will be 1171.



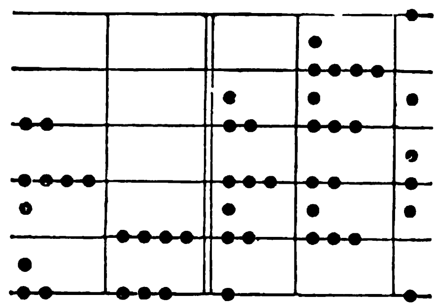
The principle of this operation is extremely simple, and the process could, with a little practice, be performed with much rapidity. In practice, the last column would not be used, as the counters on each line would be removed as the addition proceeded, and replaced by those which denoted their sum.

We will illustrate the method of subtraction by taking 682 from 1375. The two counters on the first line have none to correspond from which they can be subtracted; we therefore bring down the counter from the space above and replace it by 5 counters on the line; we shall then have 3 counters left on the line and none on the space; we then bring down 1 counter from the second space, leaving

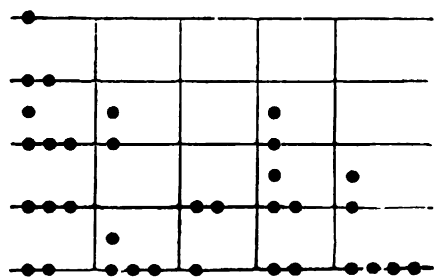


a remainder of 4 counters on the line; then bring down 1 counter from the third line to the second space, and we have 1 counter left; and so we proceed until the subtraction is complete, and we shall have as a remainder 693. Recorde writes the smaller number in the first column, and commences subtracting at the upper line.

To illustrate the process of multiplication, let us find the product of 2457 by 43. We express the multiplicand in the first column and the multiplier in the second; multiply first by 3, and place the product in the third column and the product by 4 in the fourth; add the numbers in these two columns, and the sum is the product required.

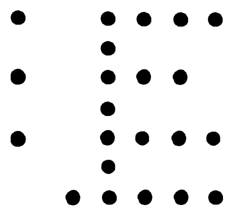


Division may be illustrated by dividing 12832 by 608. Since six hundreds is contained in 12 thousands 2 tens times, we place two counters on the second line of the quotient; multiplying 6 hundreds by 2 tens and subtracting, we have no remainder; multiplying 8 by 2 tens, we have 16 tens; but since 16 tens equal 1 hundred and 6 tens, we take off 1 from the 3 in the third or hundreds line, leaving 2 remaining; then take off 1 of those 2 and replace it by 2 in the second space, and then take 1 from the second space and 1 from the second line; then transfer the remaining counters

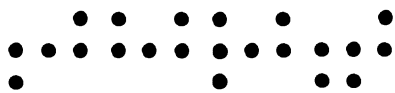


to the column of the first remainder, and we have as a remainder 672. The operation is repeated, placing the quotient 1 on the lowest line of the quotient column; and in this case we merely subtract the divisor from the first remainder, obtaining 64 for the last remainder, and 21 for the quotient. This process may evidently be repeated to any extent; but in practice it was much simplified by removing the counters of the dividend to form the first remainder, and so on until the operation was complete.

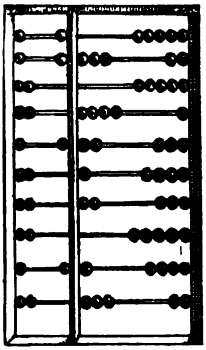
Recorde mentions two different ways of representing sums of money by means of counters, one of which he calls the *merchant's* and the other the *auditor's account*. In the margin, £198 19s. 11d. is expressed by the first method, the lowest line being pence, the second shillings, the third pounds, and the fourth scores of pounds; the spaces represent half a unit of the next superior line, and the detached counters at the left are equivalent to five counters at the right. The operations of addition, subtraction, etc. would be performed in a manner similar to those already given.



The same sum would be represented by the *auditor's account* as in the margin; the first group to the right being pence, the second shillings, the next pounds, and the left hand group scores of pounds; the two lower lines denote units of their respective classes, while in the third line those on the left denote one quarter and on the right one half of the next superior class.



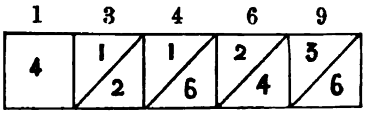
The Chinese *Computing Table* or *Swan-Pan*, previously mentioned, is represented by the accompanying engraving. It consists of a small oblong board surrounded by a frame or ledge, and parted downwards near the left side by a similar ledge. It is then divided horizontally by ten smooth and slender rods of bamboo, on which are strung two small balls of ivory or bone in the narrow compartment, and five such balls in the wider compartment; each of the latter on the several bars denoting *one*, and each of the former expressing *five*. The progressive bars, descending after the Chinese manner of writing, have their values increased ten fold at each step. The arrangement here figured denotes, reckoning downwards, the number 5,804,712,063. The *Swan-Pan* advances to the length of ten billions, or a thousand times further than the Roman Abacus. But the most admirable feature of the instrument is, that by beginning the units at any particular bar the decimal subdivisions of the unit may be represented. The Japanese make use of a similar instrument, and the facility with which they perform arithmetical operations is truly surprising.



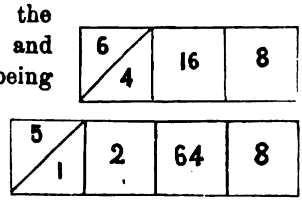
Several persons of eminence, during our own times, have advocated the revival of the practice of calculation by means of counters. Prof. Leslie considers this method as better calculated than any other to give a student a philosophical knowledge of the classification of numbers, and the theory of their notation; and he has given, in great detail, examples of the representation of numbers in different scalés of notation by counters, and of operations by means of them.

There are other species of Palpable Arithmetic, some of which have been adapted especially for the use of blind people: the celebrated Saunderson invented an instrument for this purpose with which he is said to have worked arithmetical questions with extraordinary rapidity. Arithmetical instruments of this kind possess considerable interest and importance from their use in lessening the privations consequent upon one of the greatest human calamities.

Among other arithmetical machines for shortening the work of calculation or relieving the operator from any troublesome or difficult exercise of the memory, are Napier's *virgulæ*, or *rods*, which were formerly much celebrated and generally used. The work in which they were first described was published in 1617, under the title of *Rabdologia*. In the dedication to Chancellor Seton, he says, that the great object of his life had been to shorten and simplify the business of calculation; and the invention of logarithms, which he had just promulgated, was a noble proof that he had not lived in vain. These *virgulæ*, *rods*, or *bones*, as they were often called, were thin pieces of brass, ivory, bone, or any other substance, about two inches in length and a quarter of an inch in breadth, distributed into ten sets, generally of five each; at the head of each of these, in succession, was inscribed one of the nine digits or zero, and underneath them in each piece the products of the digit at the top with each of the nine digits in succession, in a series of eight squares divided by diagonals, in the upper part of which were put the digits in the place of tens, and in the lower the digits in the place of units. In order to multiply any two numbers together, such as 3469 and 574, those rods are to be placed in contact which are headed by the digits 1, 3, 4, 6, 9, and the successive products of the terms of the multiplier into the multiplicand are found by adding successively the digit on the upper half of the square to the right to that in the lower half of the square to the left, in the line of squares which are opposite to the figure of the multiplier which is used; thus, to multiply 3469 by 4, we take the line of squares opposite 4, represented in the margin, and the product is 13876, being found by writing 6, the sum of 4 and 3, of 6 and 2, etc., carrying when necessary. In case of division, those rods are arranged in contact which are headed by the figures of the divisor, and we are thus enabled to obtain the products formed by the divisor and successive terms of the quotient.



In the case containing these rods, which Napier calls *multiplicationis promptuarium*, there are usually found also two pieces with broader faces, one consisting of three longitudinal divisions, and the other of four; one of which is adapted to the extraction of the square, and the other of the cube root; in the first, one column contains the nine digits, the second their doubles, and the third their squares; in the second, the first column contains the digits, the second their squares, and the third and fourth their cubes, two columns being necessary for this purpose when the cube consists of three places; thus, the last division but one in each of these rods is represented as in the margin, the digits occupying the right-hand column. In our times, when the multiplication table is so much more perfectly learned than formerly, the eagerness with which this invention was welcomed will excite some surprise, considering that its only object was to relieve the memory of so light and trivial a burden; but it is in accordance with some of the processes elsewhere noticed, by which early authors endeavored to simplify arithmetical operations.



Pascal, in 1642, at the age of 19, invented the first arithmetical machine, properly so called. It is said to have cost him such mental efforts as to have seriously affected his health, and even to have shortened his days. This machine was improved afterwards by other persons, but never came into practical use. In 1673, Leibnitz published a description of a machine which was much superior to that of Pascal, but more complicated in construction and too expensive for its work, since it was capable of performing only addition, subtraction multiplication and division. But these machines are entirely eclipsed by those of Babbage and Scheutz. In 1821, Mr Babbage, under the patronage of the British government, began the construction of a machine, and in 1833 a small portion of it was put together, and was found to perform its work with the utmost precision. In 1834 he commenced to design a still more powerful engine, which has not yet been constructed. The expense of these machines is enormous, $80,000 having been spent on the partial construction of the first. They are designed for the calculation of tables or series of numbers, such as tables of logarithms, sines, etc. The machine prepares a stereotype plate of the table as fast as calculated, so that no errors of the press can occur in publishing the result of its labors. Many incidental benefits have arisen from this invention, among which the most curious and valuable was the contrivance of a scheme of mechanical notation by which the connection of all parts of a machine, and the precise action of each part, at each instant of time, may be rendered visible on a diagram, thus enabling the contriver of machinery to devise modes of economizing space and time by a proper arrangement of the parts of his own invention.

A machine invented by G. and E. Scheutz, of Stockholm, and finished in 1853, was purchased for the Dudley Observatory, at Albany. The Swedish government paid $20,000 as a gratuity towards its construction. The inventors wished to attain the same ends as Mr. Babbage, but by simpler means. It can express numbers decimally or sexagesimally, and prints by the side of the table the corresponding series of numbers or arguments for which the table is calculated. It has already calculated a table of the true anomaly of Mars for each $\frac{1}{10}$ of a day. In size, it is about equal to a boudoir piano. Other attempts have been made, but so far nothing has been accomplished which is entirely satisfactory, though the utility of some such engine in the calculation of astronomical and other tables is so great, that it is quite probable that efforts will be continued until complete success is attained.

SECTION III.

ARITHMETICAL REASONING.

I. There is Reasoning in Arithmetic.

II. Nature of Arithmetical Reasoning.

III. Reasoning in the Fundamental Operations.

IV. Arithmetical Analysis.

V. The Equation in Arithmetic.

VI. Induction in Arithmetic.

CHAPTER I.

THERE IS REASONING IN ARITHMETIC.

ALL reasoning is a process of *comparison*; it consists in comparing one idea or object of thought with another. Comparison requires a standard, and this standard is the old, the axiomatic, the known. To these standards we bring the new, the theoretic, the unknown, and compare them that we may understand them. The law of correct reasoning, therefore, is to compare the new with the old, the theoretic with the axiomatic, the unknown with the known.

This process, simple as it seems, is the real process of all reasoning. We pass from idea to truth, and from lower truth to higher truth, in the endless chain of science, by the simple process of comparison. Thus the facts and phenomena of the material world are understood, the laws of nature interpreted, and the principles of science evolved. Thus we pass from the old to the new, from the simple to the complex, from the known to the unknown. Thus we discover the truths and principles of the world of matter and mind, and construct the various sciences. Comparison is the science-builder; it is the architect which erects the temples of truth, vast, symmetrical, and beautiful.

In mathematics this process is, perhaps, more clearly exhibited than in any other science. In geometry, the definitions and axioms are the standards of comparison; beginning in these, we trace our way from the simplest primary truth to the profoundest theorem. In arithmetic we have the same basis, and proceed by the same laws of logical evolution. Definitions, as a description of fundamental ideas, and axioms, as the statement of intuitive and necessary truths, are the foundation upon which we rear the superstructure of the science of numbers.

These views, though admitted in respect of geometry, have not always been fully recognized as true of arithmetic. The science, as presented in the old text-books, was simply a collection of rules for numerical operations. The pupil learned the rules and followed them, without any idea of the reason for the operation dictated. There was no thought, no deduction from principle; the pupil plodded on, like a beast of burden or an unthinking machine. There was, in fact, as the subject was presented, no science of arithmetic. We had a science of geometry, pure, exact, and beautiful, as it came from the hand of the great masters. Beginning with primary conceptions and intuitive truths, the pupil could rise step by step from the simplest axiom to the loftiest theorem; but when he turned his attention to numbers, he found no beautiful relations, no interesting logical processes, nothing but a collection of rules for adding, subtracting, calculating the cost of groceries, reckoning interest, etc. Indeed, so universal was this darkness, that the metaphysicians argued that there could be no reasoning in the science of numbers, that it is a science of intuition; and the poor pupil, not possessing the requisite intuitive power, was obliged to plod along in doubt, darkness, and disgust.

Thus things continued until the light of popular education began to spread over the land. Men of thought and genius began to teach the elements of arithmetic to young pupils; and the necessity of presenting the processes so that children could see the reason for them, began to work a change in the science of numbers. Then came the method of arithmetical analysis, in that little gem of a book by Warren Colburn. It touched the subject as with the wand of an enchantress, and it began to glow with interest and beauty. What before was dull routine, now became animated with the spirit of logic, and arithmetic was enabled to take its place beside its sister branch, geometry, in dignity as a science, and value as an educational agency.

Before entering into an explanation of the character of arithmetical reasoning, it may be interesting to notice the views of some metaphysicians who have touched upon this subject. It has been maintained, as already indicated, by some eminent logicians, that there is no reasoning in arithmetic. Mansel says, "There is no demonstration in pure arithmetic," and the same idea is held by quite a large number of metaphysicians. This opinion is drawn from a very superficial view of the subject of arithmetic,—a not uncommon fault of the metaphysician when he attempts to write upon mathematical science. The course of reasoning which led to this conclusion, is probably as follows:

First, addition and subtraction were considered the two fundamental processes of arithmetic; all other processes were regarded as the outgrowth of these, and as contained in them. Second, there is no reasoning in addition; that the sum of 2 and 3 is 5, says Whewell, is seen by intuition; hence subtraction, which is the reverse of addition, is pure intuition also; and therefore the whole science, which is contained in these two processes, is also intuitive, and involves no reasoning. This inference seems plausible, and by the metaphysicians and many others has been considered conclusive.

That this conclusion is not only incorrect but absurd, may be seen by a reference to the more difficult processes of the science. Surely, no one can maintain that there is no reasoning in the processes of *greatest common divisor*, *least common multiple*, *reduction* and *division* of fractions, *ratio* and *proportion*, etc. If these are intuitive with the logicians, it is very certain that they require a great deal of thinking on the part of the learner. These considerations are sufficient to disprove their conclusions, but do not answer their arguments; it becomes necessary, therefore, to examine the matter a little more closely.

Whether the uniting of two small numbers, as *three* and *two*, involves a process of reasoning, is a point upon which it is admitted there may be some difference of opinion. The difference of two numbers, however, may be obtained by an inference from the results of addition, and, as such, involves a process of reasoning. The elementary products of the multiplication table are not intuitive truths; they are, as will be shown in the next article, derived, as a logical inference, from the elementary sums of addition. The same is also true in the case of the elementary quotients in division. Even admitting, then, that there is no reasoning in addition or subtraction, it can clearly be shown that the derivation of the elementary results in multiplication and division does require a process of reasoning. Passing from small numbers, which may be treated independently of any notation, to large numbers expressed by the Arabic system, we see that we are required to reduce from one form to another, as from units to tens, etc., which can be done only by a comparison; and also that the methods are based upon, and derived from such general principles, as "the sum of two numbers is equal to the sum of all their parts," etc.

The great mistake, however, in their reasoning, is in assuming that all arithmetic is included in addition and subtraction. If it could be proved that addition and subtraction, and the processes growing immediately out of them, contain no reasoning, a large portion of the science remains which does not find its root in these primary processes. Several divisions of arithmetic have their origin in and grow out of comparison, and not out of addition or subtraction; and since comparison is reasoning, the divisions of arithmetic growing out of it, it is natural to suppose, involve reasoning processes. Ratio, the comparison of numbers; proportion, the comparison of ratios; the progressions, etc., certainly present pretty good examples of reasoning. These belong to the department of pure arithmetic. A proportion is essentially numerical, as is shown in another place, and belongs to arithmetic rather than to geometry. If, in geometry, the treatment of a proportion involves a reasoning process, as the logicians will surely admit, it must certainly do so when presented in arithmetic, where it really belongs. It must, therefore, be admitted that there is reasoning in pure arithmetic.

Again, if there is no reasoning in arithmetic there is no science, for science is the product of reasoning. If we admit that there is a science of numbers, there must be some reasoning in the science. And again, arithmetic and geometry are regarded as the two great co-ordinate branches of mathematics. Now it is admitted that there is reasoning in geometry, the science of extension; would it not be absurd, therefore, to suppose that there is no reasoning in arithmetic, the science of numbers?

Mansel, as already quoted, says: "Pure arithmetic contains no demonstrations." If by this he means—and I presume he does—that pure arithmetic contains no reasoning, he is answered by the previous discussion. If, however, he means that arithmetic cannot be developed in the demonstrative form of geometry—that is, by definition, axiom, proposition, and demonstration—he is also in error. Though arithmetic has never been developed in this way, it can be thus developed. The science of number will admit of as rigid and systematic a treatment as the science of extension. Some parts of the science are even now presented thus; the principles of ratio, proportion, etc., are examples. I propose, at some future time, to give a complete development of the subject, after the manner of geometry. The science, thus presented, would be a valuable addition to our academic or collegiate course, as a review of the principles of numbers. Assuming, then, that there is reasoning in arithmetic, in the next chapter I shall consider the nature of reasoning, as employed in the fundamental operations of arithmetic.

CHAPTER II.

NATURE OF ARITHMETICAL REASONING.

IN order to show the nature of the reasoning of arithmetic, a brief statement of the general nature of reasoning will be presented. All forms of reasoning deal with the two kinds of mental products, *ideas* and *truths*. An *idea* is a simple notion which may be expressed in one or more words, not forming a proposition; — as, *bird*, *triangle*, *four*, etc. A *truth* is the comparison of two or more ideas which, expressed in language, give a proposition; as, a *bird* is an *animal*, a *triangle* is a *polygon*, *four* is an *even number*. The comparison of two ideas *directly* with each other, is called a *judgment;* as, a *bird* is an *animal*, or *five* is a *prime* number. Here *five* is one idea, and a *prime number* is another idea. Judgments give rise to propositions; a proposition is a judgment expressed in words.

Nature of Reasoning. — If we compare two ideas, *not directly*, but through their relation to a third, the process is called *reasoning*. Thus, if we compare A and B, or B and C, and say A equals B or B equals C, these propositions are judgments. But if, knowing that A equals B, and B equals C, we infer that A equals C, the process is reasoning. Reasoning may, therefore, be defined *as the process of comparing two ideas through their relation to a third.* Judgment is a process of direct or immediate comparison; reasoning is a process of indirect or mediate comparison.

In thus comparing two ideas through their relation to a third, it is seen that we derive one judgment from two other judgments; hence we may also define reasoning as *the process of deriving one judgment from two other judgments;* or as *the process of deriving an unknown truth from two known truths.* The two known truths are called *premises*, and the derived truth the *conclusion;* and the three propositions together constitute a *syllogism.* The syllogism is the simplest form in which a process of reasoning can be stated. Its usual form is as follows: A equals B; but B equals C; therefore A equals C. Here "A equals B" and "B equals C" are the premises, and "A equals C" is the conclusion.

The premises in reasoning are known either by intuition, by immediate judgment, or by a previous course of reasoning. In the syllogism—"All men are mortal; Socrates is a man; therefore, Socrates is mortal"—the first premise is derived by induction, and the second by judgment. In the syllogism—"The radii of a circle are equal; R and R′ are radii of a circle; therefore R and R′ are equal"—the first premise is an intuition, and the second is a judgment. In the syllogism—"A equals B, and B equals C; therefore A equals C"—both premises are judgments.

It should also be remarked that truths drawn from the first steps of the reasoning process, do themselves become the basis of other truths, and these again the basis of others, and so on until the science is complete. This method of reasoning is called *Discursive* (*discursus*); it passes from one truth to another, like a moving from place to place. We start with the simple truths which are so evident that we cannot help seeing them; and travel from truth to truth in the pathway of science, until we reach the loftiest conceptions and the profoundest principles.

Reasoning, as we have stated, is the comparison of two ideas through their relation to a third; or it may be defined as *the derivation of one judgment from two other judgments.*

These two judgments are not always both expressed; indeed, in the usual form of thought, one is usually suppressed; but both are implied, and may be supplied if desired to show the validity of the conclusion. Every truth derived by a process of reasoning, may be shown to be an inference from two propositions which are the premises or ground of inference, and this is the test of the validity of the truth derived.

There are two kinds of reasoning, *inductive* and *deductive*. Inductive reasoning is the process of deriving a general truth from several particular ones. It is based upon the principle that *what is true of the many is true of the whole.* Thus, if we see that heat expands many metals, we infer, by induction, that it will expand all metals. Deduction is the process of deriving a particular truth from a general one. It is based upon the axiom, that *what is true of the whole is true of all the parts.* Thus, if we know, that heat will expand all metals, we infer, by deduction, that it will expand any particular metal, as iron.

Mathematics is developed by the process of deductive reasoning. The science of geometry begins with the presentation of its *ideas*, as stated in its definitions, and its self-evident *truths*, as stated in its axioms. From these it passes by the process of deduction to other truths; and then, by means of these in connection with the primary truths, proceeds to still other truths; and thus the science is unfolded. In arithmetic, no such formal presentation of definitions and axioms is made, and the truths are not presented in the logical form, as in geometry. From this it has been supposed that there is no reasoning in arithmetic. This inference, however, is incorrect; the science of numbers will admit of the same logical treatment as the science of space. There are fundamental ideas in arithmetic as in geometry; and there are also fundamental, self-evident truths, from which we may proceed by reasoning to other truths. In this chapter I shall endeavor to show the nature of the reasoning in the Fundamental Operations of Arithmetic.

Arithmetical Ideas.—The fundamental ideas of arithmetic, as given in the process of counting, are the successive numbers *one, two, three,* etc. These ideas correspond to the different ideas of geometry, and the definitions of them will correspond to the definitions of geometry. In geometry, we have the three dimensions of extension, giving us three distinct classes of ideas, *lines, surfaces,* and *volumes;* in arithmetic there is only one fundamental idea of succession, giving us but one fundamental class of notions. The primary ideas of arithmetic are *one, two, three, four, five,* etc., which correspond to the idea of *line, angle, triangle, quadrilateral, pentagon,* etc., in geometry. These ideas may be defined as in the corresponding cases in geometry. Thus *two* may be defined as *one and one; three* as *two and one,* etc.; or, in the logical form —*three* is a number consisting of two units and one unit. There are other ideas of the science growing out of relations, such as *factor, common divisor, common multiple,* etc.

Arithmetical Axioms.—The axioms of arithmetic are the self-evident truths that relate to numbers. There are two classes of axioms in arithmetic as in geometry,—those which relate to quantity in general, that is, to numbers and space; and those which belong especially to number. Thus, "Things that are equal to the same thing are equal to each other," and "If equals be added to equals the sums will be equal," etc., belong to both arithmetic and geometry. In geometry we have some axioms which do not apply to numbers, as "All right angles are equal," "A straight line is the shortest distance from one point to another," etc. There are also axioms which are peculiar to arithmetic, and which have no place in geometry. Thus, "A factor of a number is a factor of a multiple of that number," "A multiple of a number contains all the factors of that number," etc. These two classes of axioms are the foundation of the reasoning of arithmetic, as they are of the science of geometry.

Arithmetical Reasoning.—The reasoning of arithmetic is deductive. The basis of our reasoning is the definitions and axioms; that is, the conceptions of arithmetic, and the self-evident truths arising from such conceptions. The definitions present to us the special forms of quantity upon which we reason; the axioms present the laws which guide us in the reasoning process. The definitions give the subject-matter of reasoning; the axioms give the principles which determine the form of reasoning, and enable us to go forward in the discovery of new truths. Thus, having defined an angle, and a right angle, we can by comparison, prove that "the sum of the angles formed by one straight line meeting another, is equal to two right angles." Having the definition of a triangle, by comparison we can determine its properties, and the relation of its parts to each other. So in arithmetic, having defined any two numbers, as *four* and *six*, we can determine their relation and properties; or having defined least common multiple, we can obtain the least common multiple of two or more numbers, guiding our operations by the self-evident and necessary principles pertaining to the subject.

Axioms in Reasoning.—In this explanation of reasoning, it is stated that reasoning is a process of comparing two ideas through their relations to a third, and that axioms are *the laws which guide us in comparing.* This view of the nature of axioms differs from the one frequently presented. Some logicians tell us that axioms are *general truths which contain particular truths*, and that reasoning is the process of evolving these particular truths from the general ones. The axioms of a science are thus regarded as containing the entire science; if one knows the axioms of geometry, he knows the general truths in which are wrapped up all the particular truths of the science. All that is necessary for him to become a profound geometer is to analyze these axioms and take out what is contained in them.

The incorrectness, or at least inadequacy of this view of the nature of axioms and their use in reasoning, I cannot now stop to consider. Its fallacy is manifest in the extent of the assumption. It may be very pleasant for one to suppose that when he has acquired the self-evident truths of a science, he has potentially, if not actually, in his mind the entire science; such an expression may do as a figure of speech, but does not, it seems to me, express a scientific truth A *general formula* may be truly said to contain many special truths which may be derived from it; thus Lagrange's formula of Mechanics embraces the entire doctrine of the science; but no axiom can be, in the same sense, said to contain the science of arithmetic or geometry.

But whatever may be thought of this view of the nature and use of axioms, it cannot be denied that the explanation of reasoning which I have given is correct. Reasoning is the comparison of two ideas through their relation to a third, the comparison being regulated by self-evident truths. This is the view of Sir William Hamilton, and it has been adopted by several modern writers on logic. Even if the other view is right —that the axioms may be regarded as general truths, from which the particular ones are evolved by reasoning—their practical use in reasoning coincides with the explanation of the nature of the reasoning powers which I have presented; and this idea of the subject will be found to be much more readily understood and applied. The simpler view is that the *axioms are laws which guide us in the comparison*, or *they are the laws of inference.* Thus, if I wish to compare A and B: seeing that they are each equal to C, I can compare them with each other, and determine their equality by the law that *things which are equal to the same thing are equal to each other.* So, if I have two equal quantities, I may increase them equally without changing their relation, according to the law enunciated in the axiom that *if the same quantities be added to equals, the results will be equal.* This view of the subject of axioms and of their use in the process of reasoning, may be supported by various considerations, and will be found to throw light upon several things in logic upon which writers are sometimes not quite clear. In the following chapter I shall apply this view of reasoning to the fundamental operations of arithmetic.

CHAPTER III.

REASONING IN THE FUNDAMENTAL OPERATIONS.

SCIENCE, as already stated, consists of *ideas* and *truths*. Truths are derived either by intuition or reasoning. Intuitive truths come either by the intuitions of the Sense or the Reason; derivative truths by the discursive process of induction or deduction. The primary *ideas* of arithmetic are the individual numbers, *one, two, three;* its primary *truths* are the elementary sums and differences of addition and subtraction. How these primary truths are derived, is a question upon which opinion is divided. On the one hand it is claimed that they are intuitive; on the other, that they are derived by reasoning. Thus, *two* and *one* are *three*, *three* and *two* are *five*, etc., are regarded by some as *pure axioms*, neither requiring nor admitting of a demonstration; while others regard them as deductions from the primary process of counting. Let us examine the subject somewhat in detail, and also consider the process of deriving other truths growing out of these.

Addition.—It is generally assumed that the primary sums of the addition tables are axioms. They are intuitive truths growing out of an analysis of our conceptions of a number into its parts, or a synthesis of these parts to form the number. Thus, given the conception of *nine*, by analysis we see that it consists or is composed of *four* and *five;* or given *four* and *five*, by synthesis we immediately see that it gives a combination of *nine* units, or is equal to *nine*. This view is maintained by some eminent logicians. "Why is it," says Whewell, "that three and two are equal to four and one? Because if we look at five things of any kind we *see* that it is so. The five are four and one; they are also three and two. The truth of our assertion is involved in our being able to conceive the number five at all. We perceive this truth by *intuition*, for we cannot see, or imagine we see, five things, without perceiving also that the assertion above stated is true."

The other view makes counting the fundamental process, and derives the judgments expressed in the elementary sums by inference. Thus, the process of finding the sum of *five* and *four* may be stated as follows:

The sum of *five* and *four* is that number which is four units after five;
By counting we find that the number four units after five is *nine;*
Hence, the sum of *five* and *four* is *nine.*

This is a valid syllogism, and shows that the sums might be thus obtained, whether they are actually so obtained or not. It may be objected, however, that they can be obtained only in one way; and if *intuitive*, then it is not possible to derive them by any process of reasoning. This does not necessarily follow, for we can often obtain, by a process of reasoning, a truth which we could also derive in some other way. If we discover a new metal, it can be immediately inferred that heat will expand it, since heat expands all metals, which is a process of deductive reasoning. This truth may also be obtained by direct experiment. Many examples may be given to show that a truth may be derived by reasoning, which might also be derived in some other way.

These fundamental truths may be used in obtaining the relations of different combinations of numbers, and such an operation will be a process of reasoning. Thus, it is not evident to the learner, neither is it intuitive with any one, that 7 plus 2 equals 4 plus 5; or, what is less readily seen, that 25 plus 37 equals 19 plus 43. These are not axioms, since they cannot be seen to be true without an examination of the grounds of the relation. The process of reasoning to prove the propositions is as follows: 7 plus 2 equals 9; but 4 plus 5 equals 9; therefore, 7 plus 2 equals 4 plus 5; or, as Whewell puts it, thus: 7 equals 4 and 3, therefore 7 and 2 equals 4 and 3 and 2; and because 3 and 2 are 5, 7 and 2 equals 4 and 5. In the former case the result depends on the axiom, "Things that are equal to the same thing are equal to each other;" in the latter case, the reasoning process is based upon the axiom, "When equals are added to equals the results are equal." It will be noticed that Whewell's method of proof is very similar to the ordinary demonstration of the theorem that "When one straight line meets another straight line, the sum of the two angles equals two right angles."

That this is a valid process of reasoning is evident from its similarity to the geometrical process—A plus B equals C; but D plus E equals C; therefore, A plus B equals D plus E. It is readily seen that many such cases will arise in which the operations are entirely independent of the notation employed, from which it cannot be doubted that there is reasoning in addition in pure arithmetic. When we proceed to the addition of large numbers, expressed by the Arabic system, which may not be regarded as pure arithmetic, we base the operation upon the axiom that *the sum of several numbers is equal to the sum of all the parts of those numbers.* That the derivation of a result from this general axiomatic principle is a process of reasoning, cannot be doubted by any one who is competent to understand in what reasoning consists.

Subtraction.—Subtraction, like addition, embraces two cases, the finding of the difference between numbers independently of the notation employed to express them,—that is, the elementary differences of the subtraction table,—and the finding of the difference between large numbers expressed in the Arabic system. The elementary differences in subtraction may be obtained in two ways. First, we may find the difference between two numbers by *counting off* from the larger number as many units as are contained in the smaller number. Thus, if we wish to subtract *four* from *nine*, we may begin at *nine* and count backward *four* units, and find we reach *five*, and thus see that *four* from *nine* leaves *five*. The other method consists in deriving the elementary differences *by inference from the elementary sums*. The former method is regarded by some as intuitive, although it admits of a syllogistic statement; the latter method, without doubt, involves a process of reasoning.

To illustrate, suppose we wish to find the difference between *nine* and *five*. The ordinary process of thought is as follows: Since *four* added to *five* equals *nine*, *nine* diminished by *five* equals *four*. This process, put in the formal manner of the syllogism, is as follows:

The difference between two numbers is a number which added to the less will equal the greater;

But *four* added to *five*, the less, equals *nine*, the greater;

Therefore, *four* is the difference between *nine* and *five*.

This, of course, is too formal for ordinary language, but is all implied in the practical form, "*five* from *nine* leaves *four*, since *five* and *four* are *nine*." In subtracting large numbers expressed by the Arabic system of notation, we proceed upon the principle that *the difference between the parts of numbers equals the difference between the numbers themselves*, which shows that the process is one of deduction.

Multiplication.—Multiplication, like addition and subtraction, embraces two cases—the finding of the elementary products of the multiplication table, and the use of these in ascertaining the product of two numbers expressed by the Arabic system. The elementary products are obtained by deduction from the elementary sums of addition. Thus, in obtaining the product of *three times four*, the logical form of thought is as follows:

Three times *four* are the sum of *three fours;*

But the sum of three fours is *twelve;*

Hence, three times four are *twelve.*

The first premise is an immediate inference from the definition of multiplication; the second premise we know to be true from addition; the conclusion is a deductive inference from the two premises. In the common form of thought we omit one of the premises, saying, "three times four are twelve, since the sum of three fours is twelve." The multiplication of large numbers depends on these elementary products thus derived by deduction, and also employs the principle, that the sum of the products of the parts equals the whole product.

Division.—The reasoning in division is similar to that in multiplication. The elementary quotients of the division table may be obtained in two distinct ways—by *subtraction* or *reverse multiplication*, but in either case they are an inference from things already known, and are thus derived by a process of reasoning. By the method of subtraction we say, "*four* is contained in *twelve three times*, since *four* can be subtracted from *twelve* three times; by the method of reverse multiplication we say, "*four* is contained in *twelve* three times, since *three times four* are *twelve*." Each of these may be expressed in the form of a syllogism, as in multiplication. The division of larger numbers is based on these elementary quotients, and also upon the principle that the sum of the partial quotients equals the entire quotient.

The view here given concerning the origin of the elementary products and quotients may be presented in another way. When we begin addition we have no idea of multiplication; by and by the idea of a *product* arises in the mind, and it is immediately seen that *the product of the number is the sum arising from taking one number as many times as there are units in another*. Suppose then we wish to know the product of 3 times 4, we reason as follows:

The product of 3 times 4 equals the sum of 4 taken 3 times;
But the sum of 4 taken 3 times we find is 12;
Hence, the product of 3 times 4 equals 12.

Primary quotients may be obtained in a similar manner, and both are valid forms of reasoning. But whatever view may be taken of the *origin* of the elementary truths of the fundamental operations—and the fact of a difference of opinion indicates a reason for it—it certainly cannot be denied, by one who will examine, that there is reasoning in the processes growing out of these fundamental operations, and also in those which have their origin in comparison. These fundamental judgments of the tables of the four "ground rules" are committed to memory, and are employed in the reasoning processes by which we derive other truths in the science.

Other Forms.—As we leave the fundamental operations, however, the processes of reasoning grow more and more distinct. As each new *idea* is presented, new *truths* arise intuitively, which become the basis for the derivation of other truths, the same as in geometry. To illustrate, take the subject of Greatest Common Divisor. As soon as the idea of a common divisor is clearly apprehended, several truths are perceived as growing immediately out of this conception. These truths are intuitively apprehended, and are the *axioms* pertaining to the subject. From these self-evident truths, we proceed to other truths by a process of reasoning usually called demonstration. Thus, in the subject of *greatest common divisor* we have these axioms:

1. *A divisor of a number is a divisor of any number of times that number.*

2. *A common divisor of several numbers is the product of some of the common factors of these numbers.*

3. *The greatest common divisor of several numbers is the product of all the common prime factors of these numbers.*

4. *The greatest common divisor of several numbers contains no factors but those which are common to all the numbers.*

These truths are self-evident and necessary, and are seen to be so as soon as a clear idea of the subject is attained. They may be illustrated, but cannot be demonstrated. They bear precisely the same relation to the arithmetical conception of greatest common divisor that the axioms of geometry do to some of the geometrical conceptions. Thus, in geometry, as soon as we have the conception of a circle, it is intuitively seen that *all the radii are equal to each other;* or that *the radius is equal to one-half of the diameter*, etc. Such truths are made the basis of the reasoning by which we derive the other truths relating to the circle. If the process of obtaining these derivative truths in geometry is regarded as reasoning, surely the similar processes in arithmetic are also reasoning.

Having a clear conception of the *idea* of greatest common divisor, and of the self-evident *truths* or *axioms*, belonging to it, we are prepared to derive other truths relating to the subject, by the process of reasoning. As an example of a truth derived by demonstration, take the following: *The greatest common divisor of two quantities is a divisor of their sum and their difference.*

In order to demonstrate this theorem, take any two numbers, as 20 and 12. We see that the greatest common divisor is 4. We also know that 20 is 5 times 4 and 12 is 3 times 4. We then reason as follows:

The sum of the two numbers equals 5 times 4 plus 3 times 4 or 8 times 4;

But 4, the G. C. D., is evidently a divisor of 8 times 4;

Hence, 4, the G. C. D., is a divisor of the sum of the two numbers.

In this syllogism "8 times 4" is the *middle term*, the "sum of the two numbers" the *major term*, and "4, the greatest common divisor," the *minor term;* and the syllogism is entirely valid. In a similar manner we may prove that the *greatest common divisor* is a divisor of the difference of the two numbers. The method of reasoning with 20 and 12 is seen to be applicable to any two numbers having a common divisor; hence the truth is general

It should be remarked that a large portion of the reasoning in arithmetic consists in changing the form of a quantity, so that we may see a property which was concealed in a previous form, and then inferring that it belongs also to the quantity in its first form, since the value of the quantity is not changed by changing its form.

It is thus seen that the science of arithmetic, like geometry, consists of *ideas* and *truths;* that some of these truths are *self-evident*, and others are derived by a process of *reasoning;* and that the process of reasoning in the two sciences is similar. We proceed now to consider some of these forms of reasoning, and especially the subject of *arithmetical analysis*, which will be treated in the next chapter.

CHAPTER IV.

ARITHMETICAL ANALYSIS.

ARITHMETICAL Analysis is the process of developing the relation and properties of numbers by a comparison of them through their relation to the unit. All numbers consist of an aggregation of units, or are so many times the single thing; and hence bear a definite relation to the unit. This relation the mind immediately apprehends in the conception of a number itself. From this evident relation to the unit, all numbers may be readily compared with each other, and their properties and relations discovered. Let us examine the process a little more in detail.

Unit the Basis.—The basis of this analysis is the *Unit.* The Unit is the primary and fundamental idea of arithmetic. It is the basis of all numbers, a number being a repetition of the Unit, or a collection of units of the same kind. The relation of a number to the Unit, or of the Unit to a number, is consequently immediately seen from the conception of a number itself. The collection is intuitively conceived to be so many times the Unit, or the Unit such a part of the collection. The importance of the Unit, as the base of the comparison of numbers, is thus apparent. Integers may be readily compared with each other, through their relation to the fundamental elements out of which they are formed.

A Unit is *one* of the several things considered; and, since a fraction is a number of equal parts of a Unit, it is seen that we have a second class of units which we may call *fractional* *units.* These two classes of units may be distinguished as the *Unit* and the *fractional unit.* A number of fractional units gives rise to a class of numbers called *fractions.* The same principle of comparison obtains in the comparison of these as in the comparison of integral numbers. A *fractional unit* being *one* of several equal parts of the Unit, its relation to the latter is simple and immediately apprehended. We can thus compare different *fractional units* by their relation to the *Unit,* as we did integral numbers by their relation to it. The comparison of fractions, which at first might have seemed difficult, thus becomes simple and easy.

From this consideration we are enabled to see the importance of the Unit in the process of arithmetical analysis. As the basis of numbers, it becomes the basis of reasoning with numbers. We compare number with number or fraction with fraction by their intermediate relation to the Unit. The Unit thus becomes the stepping-stone of the reasoning process, the central point around which the circle of logic revolves.

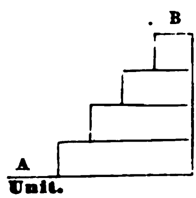
Comparison of Integers.—Numbers are compared, as has already been remarked, by their relation to the Unit. In the comparison of numbers, the relation between them is not immediately apprehended; but knowing the relation that each sustains to the Unit, we can ascertain their relation to each other by this simple intermediate relation. To illustrate this, suppose we wish to compare any two numbers, as 3 and 5; let the problem be "What is the relation of 3 to 5?" or "3 is what part of 5?" We would reason thus: *One* is 1 fifth of 5, and if *one* is 1 fifth of 5, 3, which is *three* times *one,* is *three* times 1 *fifth,* or 3 *fifths* of 5. Hence, 3 is 3 fifths of 5. In this example we cannot compare 3 directly with 5; we therefore make the comparison indirectly, by considering their intermediate relation to the Unit, which is readily apprehended. Again, take the problem, "If 3 times a number is 12, what is 5 times the number?" Here, it may be remarked, 3 *times* the *number* is the *known* quantity, and 5 *times* the *number* is the *unknown* quantity, which we wish to find by comparing it with the known quantity. How shall we make this comparison, and thus pass from the known to the unknown? We cannot compare them *directly*, since the relation between them is not readily perceived; we must compare them *indirectly* by means of their relation to the Unit. The process of reasoning is as follows: If 3 times a number is 12, *once* the number is $\frac{1}{3}$ of 12 or 4; and if *once* a number is 4, *five* times the number is 5 times 4, or 20. Thus we readily pass from three times the number to five times the number—from the known to the unknown—first passing from *three* to *one* and then from *one* to *five*. In the same manner all numbers may be compared with each other, their relation being determined by this intermediate relation to One, the Unit, the basis of all numbers.

Comparison of Fractions.—Fractions are also compared by means of their relation to the Unit. A Fraction is a *number of fractional units*. The fractional unit is one of several equal parts of the Unit; hence the relation between it and the Unit is simple and readily perceived. When we have a number of fractional units—that is, a Fraction—in comparing it with the Unit, we must first pass from the *number* of fractional units to the *fractional unit* itself, and then from the *fractional unit* to the *Unit*. From this we can readily pass to a *number*, or to any other *fractional unit*, and then to any number of such fractional units, that is, to any fraction. This will be more clearly seen by its application to a problem.

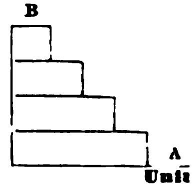
Take the problem, "If $\frac{2}{3}$ of a number is 24, what is $\frac{3}{4}$ of the number?" We reason thus: If *two*-thirds of a number is 24, *one*-third of the number is $\frac{1}{2}$ of 24, or 12; and *three*-thirds, or *once* the number, is 3 times 12, or 36. If *once* the number is 36, *one*-fourth of the number is $\frac{1}{4}$ of 36, or 9; and *three*-fourths of the number is 3 times 9, or 27. In this problem we compare the two fractions $\frac{2}{3}$ and $\frac{3}{4}$, by passing from *two*-thirds down to *one*-third, then rising up to the Unit, then passing down to *one*-fourth, and then up to *three*-fourths. In other words, we pass from a *number* of fractional units to the *fractional unit*, then to the *Unit*, then to another *fractional unit*, and then to a *number* of those fractional units. We first go down, then up, then down again, and then up again to the required point.

Another excellent example of this method of comparison is given in the solution of the following problem: What is the relation of $\frac{2}{3}$ to $\frac{4}{5}$? Here $\frac{4}{5}$ is the basis of comparison with which it is required to compare $\frac{2}{3}$. This relation cannot be immediately seen, but it can readily be determined by the method of analysis. The solution is as follows: *One*-fifth is $\frac{1}{4}$ of $\frac{4}{5}$, and if *one*-fifth is $\frac{1}{4}$ of $\frac{4}{5}$, *five*-fifths, or *One*, is 5 times $\frac{1}{4}$ or $\frac{5}{4}$ of $\frac{4}{5}$. If *One* is $\frac{5}{4}$ of $\frac{4}{5}$, *one*-third is $\frac{1}{3}$ of $\frac{5}{4}$ or $\frac{5}{12}$ of $\frac{4}{5}$, and *two*-thirds is 2 times $\frac{5}{12}$, or $\frac{5}{6}$; hence $\frac{2}{3}$ is $\frac{5}{6}$ of $\frac{4}{5}$. In this problem we see the same law of comparison, and this law runs through the entire subject.

Having given this general idea of the process, I will state the several simple cases of arithmetical analysis, and illustrate the process of thought by means of a diagram. The central relation of the Unit to the thought process, and the transition *from* the Unit and *to* the Unit, will be readily seen.

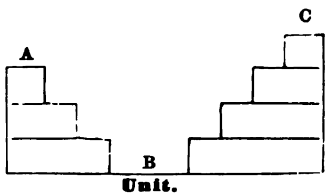


CASE I.—*To pass from the Unit to any number.* Take the problem: If 1 apple costs 3 cents, what will 4 apples cost? If 1 apple costs 3 cents, 4 apples, which are 4 times 1 apple, will cost 4 times 3 cents, or 12 cents. In this problem the mind starts at the Unit A, and ascends 4 steps to B.



CASE II.—*To pass from any number to the Unit.* Take the problem: If 4 apples cost 12 cents, what will 1 apple cost? The solution is as follows: If 4 apples cost 12 cents, 1 apple, which is 1 fourth of 4 apples, will cost 1 fourth of 12 cents, or 3 cents. In this problem the mind starts at the number 4, four steps above the basis, and steps down to the Unit, or basis of numbers.

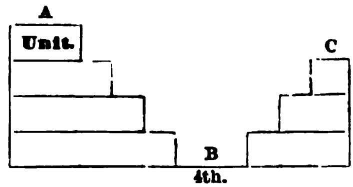
Case III.—*To pass from a number to a number.* Take the problem: If 3 apples cost 15 cents, what will 4 apples cost? The solution is: If 3 apples cost 15 cents, 1 apple will cost $\frac{1}{3}$ of 15 cents, or 5 cents, and 4 apples will cost 4 times 5 cents, or 20 cents. In this case we are to pass from the collection *three* to the collection *four*. In comparing *three* and *four*, their relation is not readily seen; but knowing the relation of *three* to the *Unit*, and of the *Unit* to *four*, we make the transition from *three* to *four* by passing through the *Unit*. This may be illustrated as follows: Suppose one standing at A and wishing to pass over to C. Unable to step directly from A to C, he first steps down to the starting point, B, and then ascends to C. So in comparing numbers, when we cannot pass directly from the one to the other, we go down to the Unit, or starting-point of numbers, and then go up to the other number. These relations are intuitively apprehended, being presented in the formation of numbers. In the given problem we stand *three* steps above the Unit, and we wish to go *four* steps above the Unit. To do this we first *descend* the *three* steps, and then *ascend* the *four* steps.



Case IV.—*To pass from a unit to a fraction.* Take the problem: If one ton of hay cost $8, what will $\frac{3}{4}$ of a ton cost? The solution is as follows: If *one* ton of hay costs $8, one-fourth of a ton will cost $\frac{1}{4}$ of $8, or $2, and three-fourths of a ton will cost 3 times $2, or $6.

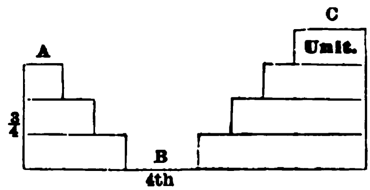
In this problem we pass from the *Unit* to the *fourth*, one of the equal divisions of the unit, and then to a collection of such equal divisions. In other words, we *descend* from the integral

Unit to the fractional Unit, and then *ascend* among the fractional units. It is as if we were standing at A, and wished to pass to C; we first take *four* steps down to B, and then *three* steps up to C, instead of trying to step immediately over from A to C.



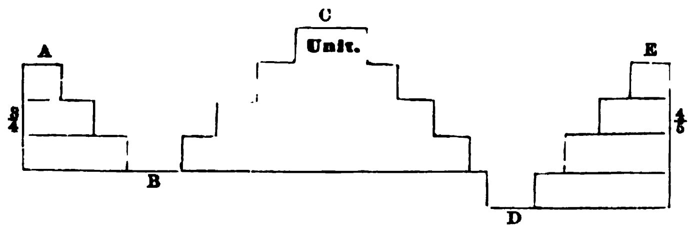
Case V.—*To pass from a fraction to a unit.* Take the problem: If $\frac{3}{4}$ of a ton of hay cost \$6, what will one ton cost? The solution is as follows: If *three*-fourths of a ton of hay cost \$6, *one*-fourth of a ton will cost $\frac{1}{3}$ of \$6, or \$2; and *four*-fourths of a ton, or *one* ton, will cost 4 times \$2, or \$8.

In this problem we pass from a *collection* of fractional units to the *fractional unit*, and then to the integral Unit. It is as if we were standing at A, and wished to pass to C. We cannot make the transition directly, so we step *three* steps down to B, and then *four* steps up to C.



Case VI.—*To pass from a fraction to a fraction.*—Take the problem· If $\frac{3}{4}$ of a number is 15, what is $\frac{4}{5}$ of the number? The solution is as follows: If *three*-fourths of a number is 15, *one*-fourth of the number is $\frac{1}{3}$ of 15 or 5, and *four*-fourths of the number, or *once* the number, is 4 times 5 or 20. If the number is 20, *one-fifth* of the number is 1 fifth of 20, or 4; and *four*-fifths of the number is 4 times 4 or 16.

In this problem we wish to compare the two fractions $\frac{3}{4}$ and $\frac{4}{5}$; but since we cannot perceive the relation of them directly, we must compare them through their relation to the Unit. To do this we first go from *three*-fourths to *one*-fourth, then from *one*-fourth to the Unit, then from the Unit to *one-fifth*, and then to *four-fifths*. In other words, we first go down from the *collection* of fractional units to the fractional *Unit* and then up to the *integral Unit;* we then descend to the *other fractional unit*, and then ascend to the *number of fractional units* required. It is as if we were standing at A and wished to pass to E; we cannot step directly over from one point to the



other so we pass from A *three* steps down to B, then *four* steps up to C, then *five* steps down to D, and then *four* steps up to E.

These diagrams, it is believed, present a clear illustration of the subject, and enable one to understand the process of thought in the elementary operations of arithmetical analysis. The Unit is thus seen to lie at the basis of the process, the mind running to it and from it in the comparison of numbers. It will be remembered, however, that these are merely illustrations, and are not designed to convey a complete idea of the process in all of its details. This can only be seen by a careful analysis of the process itself.

Analysis Syllogistic.—The process of arithmetical analysis is a process of mediate comparison, and is consequently a reasoning process. This will appear from the fact that it may be presented in the syllogistic form. Take the simplest case: If 4 apples cost 12 cents, what will 5 apples cost? Expressed in the form of a syllogism, we have the following:

The cost of 1 apple is $\frac{1}{4}$ of the cost of 4 apples;
But $\frac{1}{4}$ of the cost of 4 apples is $\frac{1}{4}$ of 12 cents, or 3 cents;
Hence the cost of 1 apple is 3 cents.

The cost of *five* apples is 5 times the cost of 1 apple;
But, 5 times the cost of 1 apple is 5 times 3 cents, or 15 cents;
Hence, the cost of five apples is 15 cents.

It is thus seen that the process of analysis is purely syllogistic, and is, consequently, a reasoning process. It is not usually presented in the syllogistic form, since it would be too stiff and formal, and moreover would be more difficult for the young pupil to understand.

Direct Comparison.—The comparison of numbers, so far as explained, is indirect and mediate, that is, through their relation to the Unit. After becoming familiar with this process, the mind begins to perceive the relations between numbers themselves, and is thus enabled to reason by comparing the numbers directly, instead of employing their intermediate relations to the common basis. To illustrate, take the problem: If 3 apples cost 10 cents, what will 6 apples cost? We may reason thus: If 3 apples cost 10 cents, 6 apples, which are *two times* 3 apples, will cost *two times* 10 cents, or 20 cents. Primarily we would have gone to the Unit, finding the cost of *one* apple; but now we may omit this and compare the numbers directly.

With integral numbers this direct comparison is simple and easy; but with fractions it is much more complicated and difficult. Thus, if $\frac{2}{3}$ of a number is 20, it is difficult to see directly that $\frac{4}{5}$ of the number is $\frac{6}{5}$ of 20; that is, that the relation of $\frac{4}{5}$ to $\frac{2}{3}$ is $\frac{6}{5}$; hence, though we should avail ourselves of the direct relation of integral numbers, it will be found much simpler to compare fractions by their intermediate relations to the Unit.

CHAPTER V.

THE EQUATION IN ARITHMETIC.

THE comparison of mathematical quantities is mainly concerned with the relations of equality. The relation of equality gives rise to the *Equation*, one of the most important instruments of mathematical investigation. The *Equation* lies at the basis of mathematical reasoning; it is the key with which we unlock its most hidden principles; the instrument with which we develop its profoundest truths. The equation is a universal form of thought, and is not restricted to any one branch of mathematics. In its simple form it belongs to arithmetic and geometry, as well as to algebra. The simplest process of arithmetic, *one and one are two* ($1+1=2$), is really an equation, as much as $x^2+ax=b$.

In the higher departments of the subject of arithmetic, the equational form of thought and expression becomes indispensable. Much of the reasoning of arithmetic, which is not formally thus expressed, may be put in the form of the equation. As an example, take the question, "If $\frac{2}{3}$ of a number is 24, what is the number?" The solution of this may be expressed as follows: Since $\frac{2}{3}$ of the number = 24, $\frac{1}{3}$ of the number = 12, and $\frac{3}{3}$ of the number=36. Here, "the number" is the *unknown* quantity, which is ascertained by comparing it with the *known* quantity, 24; and then, by the analysis, passing from *two-thirds* of the number to *once* the number. The illustration given is of a very simple case, but the same principle holds in the most complicated processes of arithmetical analysis. If, instead of *a number*, we had *value, cost, weight, labor*, etc., the method of comparison and analysis would be the same. We can thus see the use of the equation, the great instrument of analysis, even in the elementary processes of arithmetic. Here it begins that wondrous career which ends in the deepest analysis and the broadest generalization. Here we find the germ of that power which, in its higher development, comprehends the whole science of Mechanics in a single formula, thus holding, potentially, in its mighty grasp, the mathematical laws of the universe.

The equation in arithmetic assumes several different forms. We begin by comparing quantities—the comparison of equal quantities giving an equation. A comparison of unequal quantities gives us *ratio*, and a comparison of equal ratios gives us another kind of equation, an equation of relations, usually called a *proportion*. The proportion 4:2::6:3, is in reality an equation, as much so as $2=2$, for it really means $4\div2=6\div3$. The treatment of the equation gives rise to several special forms of logical procedure, such as *transposition, elimination*, etc.

The equation, I have said, belongs to arithmetic; and this thought I desire to impress. The equation is a formal comparison of two equal quantities. This comparison is being made continually; all of our reasoning involves it; we cannot think without it; hence, the equation must enter into the reasoning of arithmetic. We compare one thing with another, the known with the unknown, and thus attain to new truths; and all such forms of comparison involve the equation, and are only possible by means of it. The simplest arithmetical process, $1+1=2$, is as much an equation as $Du=\delta u+du$, though the latter may express one of the profoundest generalizations to which the human mind has attained.

Substitution.—A prominent element of arithmetical reasoning, accompanying the equation, is *substitution*. By this we mean the using of one quantity in place of another, to which it is equal. The object of this is that if we have an expression consisting of a combination of several different quantities, and know the relation of these quantities, we may so substitute their values that the expression for the combination may be obtained in terms of one single quantity, the value of which may much more readily be determined; and then the values of the other quantities, from their relation to this quantity, may also be found.

To illustrate, suppose we have the two conditions, *twice a number plus three times another number equals* 48, and *three times* this second number equals *four times* the first. We can readily solve this by substituting for one of these numbers its value in terms of the other, thus obtaining a number of times a single quantity, equal to the known quantity 48. The operation may be exhibited thus:

2 times the first number + 3 times the second = 48;
but, 3 times the second number = 4 times the first number;
hence, 2 times the first number + 4 times the first number = 48;
or, 6 times the first number = 48,
and, *once* the first number = 8,
and from this we may easily find the second number.

Substitution is a form of deductive reasoning, as may be seen by an analysis of the process. Take the simple example, $A+B=24$, and $B=3A$. We usually reason as follows: If $A+B=24$, and $B=3A$, then $A+3A=24$, or $4A=24$, etc. That the logical character of the process may appear, we should reason thus: If $B=3A$, $A+B$ will equal $A+3A$, from the axiom, "If equals be added to equals the sums will be equal." And since $A+B=24$, and $A+B=A+3A$, $A+3A$ must equal 24, from the axiom, "Things that are equal to the same thing are equal to each other." Substitution is thus seen to be strictly a deductive process. In practice these logical steps are omitted for brevity and conciseness, the argument being sufficiently clear to be readily understood.

Substitution is almost an essential accompaniment of the equation. The comparison of two equal quantities without some other truth, would often be of little value in attaining new truth. By substituting one value for another, we can often so change the equation that it expresses a relation which will immediately lead to some new relation of the known to the unknown, by which we can attain to the value of the unknown. Substitution has been supposed to be restricted to algebraic reasoning; but this is not correct. It is extensively employed in geometrical reasoning, and is just as appropriate in arithmetic as in algebra.

Transposition.—In the equational form of thought, so constantly recurring in arithmetic, it sometimes occurs that we have a multiple of a quantity compared with another multiple of the same quantity, increased or diminished by some other quantity. In such cases it is natural to desire to unite these two multiples into one, which is done by so changing them as to bring them on the same side of the equation. This is what is known as *transposition.* It is consequently seen that transposition is a process not foreign to arithmetic, but one entirely legitimate and natural in the comparison of arithmetical ideas.

Other processes of thought analogous to those which occur in algebra are employed in arithmetical reasoning. The mind here takes the first step in equational thought, which, when generalized, leads it to the high altitudes of mathematical science. Here it plumes its wings to follow the master minds in their lofty flights in a region of thought far beyond that of which the mere arithmetician could even dream. The object of this chapter is not to give a philosophical discussion of the equation in general, but to show that it has a place even in arithmetical reasoning, which has sometimes been doubted or denied.

CHAPTER VI.

INDUCTION IN ARITHMETIC.

MATHEMATICS is a deductive science, and all of its truths, not axiomatic, may be derived by a deductive process of reasoning. Is it possible, however, to obtain any of these truths by Induction? This is a disputed question; it will therefore, it is thought, be of interest to enter somewhat into details in its discussion. I believe it can be shown that there are many truths in mathematics that can be proved by induction; and, furthermore, that many of its truths were originally obtained by an inductive process; and still further, that induction is, in many cases, a legitimate method of mathematical investigation.

Induction, as is generally known, is a process of thought from particular facts and truths to general ones. It is the logical process of inferring a general truth from particular facts or truths. Thus, if I observe that heat will expand the several metals, iron, tin, zinc, lead, etc., I may infer, since these are representatives of the class of metals, that heat will expand all metals. It is thus seen to be a process of reasoning, based upon the principle that what is true of the individuals is true of the class. The basis of Induction is the general proposition that what is true of the many is true of the whole; or, as Esser states it, "What belongs or does not belong to many things of the same kind belongs or does not belong to all things of the same kind."

That this method of reasoning can be employed in arithmetic appears evident *a priori.* It is certainly not unreasonable to suppose that we may, upon finding a truth which holds in several particular cases in arithmetic, infer that it will hold good in all similar cases. This conclusion is strengthened by the fact that arithmetic is somewhat special in its nature, particularly so as compared with algebra. Its symbols represent special numbers, and dealing thus with special symbols, it is to be expected that we would discover some truths which hold in particular instances, before we know of their general application. That it is not only possible to reason inductively in arithmetic, but that we do reason thus, may be shown by actual examples.

First, take the property of the divisibility of numbers by *nine.* Suppose that, not knowing this property, I divide a number by 9, and then divide the sum of the digits by 9, and thus see that both remainders are the same. Suppose I should try this with several different numbers, and seeing that it holds good in each case, infer that it is true in all cases; should I not have entire faith in my conclusion, and would not this inference be well founded? This is an inductive inference, and is as legitimate as the inference that heat expands all metals, because we see that it expands the several particular metals, iron, zinc, tin, etc.

Second, take a number of two digits, as 37; invert the digits, and take the difference between the two numbers, and we have 73—37 equal to 36, in which the sum of the two digits, 3 and 6, equals 9. If we take several other numbers of two digits and do the same, we shall find the sum of the two digits to be also 9; and observing that this is true in several cases, we may infer that it is true in all cases, in which we again have a true inductive inference.

Third, take a proportion in arithmetic, and, by actual multiplication, we shall see that *the product of the means equals the product of the extremes.* Examining several proportions, we shall see that the same is true in each case, and from these we can infer that it is true in all cases, in which we again arrive at a general truth by induction. This is not only legitimate inference, but it is actually the way in which pupils naturally derive the truth before they understand how to demonstrate it.

Now, of course each of the above principles will admit of rigorous demonstration by deduction; what I hold, and what I think is clearly shown, is, that they can also be derived by induction. Deduction would prove that they must be so; induction merely shows that they are so. Many other examples from arithmetic might be given in illustration of the same thing. But the use of induction in mathematics is not confined to arithmetic; if we go to algebra we shall find that the same method of reasoning may be, and indeed is, employed there. The theorem, $x^n - y^n$ is divisible by $x - y$, may be proved by pure induction. Try the several cases $x^2 - y^2$, $x^3 - y^3$, $x^4 - y^4$, etc., and seeing that the division is exact in the several cases, it is entirely legitimate to infer that it will be exact in all similar cases, or that $x^n - y^n$ is divisible by $x - y$. The same thing may be shown in many other cases, but it is needless to multiply examples. Even in geometry the same method may be applied. I knew a young person who, before he studied geometry, derived by trial and induction the fact that there may be a series of right-angled triangles, whose sides are in the proportion of 3, 4, and 5; and there is no doubt that the ancients knew that the square of the hypothenuse equaled the sum of the squares on the other two sides, long before Pythagoras demonstrated it.

I have said that some of the truths of mathematics were discovered by induction; among these the most prominent, perhaps, is Newton's Binomial Theorem. Newton discovered this theorem by pure induction. He left no demonstration of it, and yet it was considered of so much importance that it was engraved upon his tomb. His first principles of Calculus were somewhat inductive in their origin, as may be seen in his Principia.

The following formula is used for finding the number of primes up to the number x, when x is a large number:

$$N=\frac{x}{A \log x-B};$$

in which N denotes the number of primes, and A and B are constants to be determined by trial. This formula was derived by a process of induction. It is found to satisfy the tables of prime numbers, but no deductive demonstration of it has yet been given, and it must therefore be regarded as empirical.

In the theory of numbers we have the following remarkable property: *Every number is the sum of one, two, or three triangular numbers; the sum of one, two, three, or four square numbers; the sum of one, two, three, four, or five pentagonal numbers, and so on.* This law, though known to be entirely general, has never been demonstrated except for the triangular and square numbers. It was discovered by Fermat, who intimates, in his notes on Diophantus, that he was in possession of a demonstration of it; which, however, is doubtful, since such mathematicians as Lagrange, Legendre, and Gauss have failed to demonstrate it. The general law is at present accepted on the basis of induction.

It is thus clearly seen that many of the truths of mathematics can be derived by induction; that is, by inferring general truths from particular cases. It is not claimed, however, that this changes the nature of the science. I have before said that mathematics is a deductive science; my object has been merely to show the error of those who hold that it is impossible to derive any of the truths of mathematics by induction.

I have called especial attention to this subject, on account of the obscure and conflicting views which seem to exist concerning it. Several authors speak of the inductive methods of treating arithmetic, while others as positively assert that there can be no inductive treatment of the science. The logicians lead us to infer that induction cannot be applied to mathematics, and not a few of them distinctly assert it. Dr. Whewell says, in speaking of mathematics: "These sciences have . . . . no process of proof but deduction." Prof. Dodd wrote several pamphlets to prove that there can be no such thing as inductive reasoning in arithmetic; and several of those whom he criticised in these articles, have acknowledged the correctness of his views, and consequently, their own mistakes.

These views, I have already shown, are only partially true. Arithmetic is a deductive science; all of its truths may probably be derived by deduction; but it is equally true that some of them may also be obtained by induction, as has been shown above; and also, that some of them are accepted alone on induction, having never been demonstrated.

Great care should be exercised, however, in the use of induction in mathematics. Several supposed truths which were derived by induction were subsequently found to be untrue. Fermat asserted that the formula, 2^m+1 is always a prime, when m is taken any term in the series 1, 2, 4, 8, 16, etc., but Euler found that $2^{32}+1$ is a composite number. Lagrange tells us that Euler found by induction the following rule for determining the resolvability of every equation of the form $x^2+Ay=B$, when B is a prime number: the equation must be possible when B shall have the form, $4An+r^2$, or $4An+r^2-A$. This proposition holds good for a large number of cases, and was thought by many mathematicians to be entirely general, but the equation, $x^2-79y^2=101$, Lagrange proves to be an exception to it.

The danger of inductive inference in mathematics is also seen in some of the formulas which have been presented for finding prime numbers. Several of these hold good for many terms, and were supposed to be general, but were at last found to be only special. Thus, the formula x^2+x+41 holds good for *forty* values of x. The formula x^2+x+17 gives *seventeen* of its first values prime, and $2x^2+29$ gives *twenty-nine* of its first values prime.

Having shown that mathematics, though a deductive science, will admit, in some instances, of an inductive treatment, it may be remarked that such treatment is especially adapted to young pupils in the elementary processes of arithmetic. It is difficult for them to draw conclusions from the principles established by a deductive demonstration; hence, in some cases, it may be well for them to employ the inductive method. The rules for working fractions may be derived by an inductive inference from the solution of a particular example; and this method will be much more readily understood than the derivation of them from general principles deductively established. The method is to solve a particular problem by analysis, and then derive a general method by an inductive inference from such analysis. Thus analysis and induction become, as it were, golden keys with which we unlock the complex combinations of numbers.

It will be well, however, to lead the pupils to the deductive method as soon as possible. Most students will make the transition naturally. The better reasoners among them will themselves rise from this inductive method, being satisfied only with a deductive demonstration; and in this they should be encouraged. They will often see the deductive, or necessary idea, behind the inductive process, and thus pass spontaneously from the particular fact to the general truth. They will sometimes discover a truth by trial and inference, that is, by induction, and then learn to demonstrate it deductively; and it will be a useful exercise for pupils to have some special drill in this manner. They will thus see the relation of the two methods of reasoning, and be impressed with the deductive nature of the science of arithmetic, and the necessary character of its truths.

Part II.

SYNTHESIS AND ANALYSIS.

SECTION I.

FUNDAMENTAL OPERATIONS.

SECTION II.

DERIVATIVE OPERATIONS.

SECTION I.

FUNDAMENTAL OPERATIONS OF SYNTHESIS AND ANALYSIS

I. Addition.

II. Subtraction.

III. Multiplication

IV. Division.

CHAPTER I.

ADDITION.

THE fundamental synthetic process of arithmetic is *Addition.* Beginning at the Unit as the primary numerical idea, numbers arise by a process of synthesis. By it we pass from unity to plurality; from the one to the many. This mental process which gives rise to numbers, we naturally extend to the numbers themselves, and thus synthesis becomes the primary operation of arithmetic. This general synthetic process is called Addition.

Definition.—Addition is the process of finding the *sum* of two or more numbers. The *sum* of two or more numbers is a single number which expresses as many units as the several numbers added. The sum is often called the *amount.*

Addition may also be defined as the process of uniting several numbers into one number which expresses as many units as the several numbers united. This last definition includes both of the previous ones, and avoids the use of the word *sum.* The former definition is, however, preferred on account of its conciseness and simplicity, and is the one usually adopted by arithmeticians.

Principles. — The process of addition is performed in accordance with certain necessary laws which are called principles. The most important of these are the following:

I. *Only similar numbers can be added.* Thus, we cannot find the sum of 4 apples and 5 peaches, for if we unite the numbers we shall have neither 9 apples nor 9 peaches. It has been claimed, that the sum is 9 *apples and peaches;* in proof of which it is said we speak properly of "12 knives and forks," meaning 6 knives and 6 forks. Such a combination is, however, popular rather than scientific; it is not what we mean by a strict use of the word addition.

It may also be observed that dissimilar numbers may be brought under the *same name* and thus become similar, when they can be united in one sum. Thus, 4 sticks and 5 stones may be regarded as so many *objects* or *things*, and their sum will be 9 *objects* or 9 *things.* So in writing units and tens in the Arabic system; they cannot be combined directly, but by reducing both to *tens* or both to *units*, the addition can be effected.

II. *The sum is a number similar to the numbers added.* This is evidently an axiomatic truth. The sum of 4 *cows* and 5 *cows* is 9 *cows*, and cannot be *horses* or *sheep*, or anything besides cows. An apparent exception which will be understood by what is said above is, that the sum of 3 *horses* and 5 *cows* is 8 *animals.*

III. *The sum is the same in whatever order the numbers are added.* This is evident from the consideration that in any case we have the combination of the same number of units, and consequently the same sum.

Cases.—Addition is divided philosophically into two general cases. The first case consists in finding the sums of numbers independently of the notation used to express them. The second case consists in finding the sum of numbers as expressed in written characters, and thus grows out of the use of the Arabic system of notation. The former deals with small numbers which can be united mentally, and may be called *mental addition;* the latter is used with large numbers as expressed with written characters, and may be called *written addition.* The former is a process of *pure arithmetic;* the latter is incidental to the system of notation which may be employed, and is not essential to number in itself considered.

The former method is an independent process, complete in itself; the latter is dependent upon the former for the elements with which it works. By the former case we obtain what we may call the *primary sums* of addition, or what is generally known as the *Addition Table*, which we make use of in adding large numbers expressed by the Arabic method of notation.

Treatment.—The primary synthetic arithmetical process is that of increasing by units. This process is presented in the genesis of numbers where, by counting, we pass from one number to another immediately following it, by the addition of a unit; and it also lies at the foundation of the method by which we find the sum of any two or more numbers. By it we obtain the elementary sums of the first case, and then we use these sums in solving the problems of the second case. The method of treating both of these cases will be presented somewhat in detail.

CASE I. *To find the primary sums of arithmetic.*—The primary sums of arithmetic are found by the same process of counting by which our ideas of numbers are generated. The sum of two numbers is primarily determined by beginning at one number and counting forward from it as many units as are in the number to be added to it. Thus, to find the sum of any two numbers, as *five* and *four*, we begin at *five* and count four successive numbers, *six*, *seven*, *eight*, *nine*, and seeing we reach *nine*, we know that *five* and *four* are *nine*. In this way we obtain the sums of all small numbers, and then *commit them to memory*, that we may know them when we wish to use them without passing through the steps by which they were obtained.

To be assured that this is the real method, we have but to watch young children when adding, and we shall see that they do actually find the sums of numbers in the manner explained. They may often be seen counting their fingers, or marks on the slate, in performing addition. The elementary sums thus found are the basis of addition. We fix them in the memory as we do the elementary products of the multiplication table, and employ them in finding the sums of larger numbers.

These primary sums may be regarded as the *axioms* of addition. They are intuitive truths, that is, truths which cannot be demonstrated, but are seen by intuition. "Why is it," says Whewell, "that three and two are equal to four and one? Because if we look at five things of any kind, we *see* that it is so. The *five* are *four* and *one;* they are also *three* and *two* The truth of our assertion is involved in our being able to conceive the number *five* at all. We perceive this truth by *intuition*, for we cannot see, or imagine we see, five things, without perceiving also that the assertion above stated is true."

CASE II. *To add numbers expressed by the Arabic system of notation.*—The principle by which we find the sum of larger numbers expressed by the Arabic system, is that of *adding by parts.* Having learned the sums of small numbers, we separate larger numbers into parts corresponding to these small numbers, and then find the sum of these parts which, united, will give the entire sum. Thus in practice we first add the units group, then the tens group, and thus continue until all the groups are added. If the sum of any group amounts to more than nine units of that group, we incorporate the *tens* term of the sum with the sum of the next higher group.

Solution.—Thus, in adding the two numbers 368 and 579, we write the numbers so that similar terms stand in the same column, and begin at the right to add. 9 units and 8 units are 17 units, or 1 ten and 7 units; we write the 7 units, and add the 1 ten to the sum of the next column. 7 tens and 6 tens are 13 tens, and 1 ten are 14 tens, or 1 hundred and 4 tens; we write the 4 tens, and add the 1 hundred to the next column. 5 hundreds and 3 hundreds are 8 hundreds, and 1 hundred are 9 hundreds, which we write in hundreds place The entire sum is therefore 947.

OPERATION.

$$\begin{array}{r} 368 \\ 579 \\ \hline 947 \end{array}$$

This method of adding by parts is the result of the beautiful system of Arabic notation, whereby figures in different positions express groups of different value. It is peculiar to this method of expressing numbers, and illustrates its great convenience and utility. In adding large numbers, it would be exceedingly difficult, if not impossible, for the mind to unite them directly into one sum; but by adding the groups separately, the process is simple and easy.

Rule.—One of the most common errors of arithmetic is found in the statement of the rules of the fundamental operations. This error consists in confounding the meaning of the words *figure* and *number*. Thus, it is usual to speak of "adding the figures," of "carrying the left-hand figure to the next column," etc. This is a mistake involving a looseness of thought that ought not to be permitted to remain in the text-books. We cannot add *figures*, we can add only the *numbers* which they express.

This error can be avoided in several ways. The method here suggested is the use of the word *term* for figure. The word term is already employed in a similar manner in algebra. It may be used in a dual sense, embracing both the figure and the number expressed by the figure. Numbers and figures have a definite signification, and one cannot be used for the other without a mistake; but it will be both correct and convenient to use one word for both. No ambiguity will be occasioned by it, as the particular meaning may be determined by the application. In this way we may avoid the error of speaking of "adding figures," and also the inconvenient expression sometimes employed of "adding the numbers denoted by the figures."

Why do we write the numbers as suggested, and why do we begin at the right hand to add, are questions very frequently asked of the arithmetician. In adding numbers we write them one under another, so that figures of the same order stand in the same vertical column, for convenience in adding. We begin at the right hand to add as a matter of convenience also, so that when the sum of any column exceeds nine units of that column, we may unite the number denoted by the left hand term to the next column. We can also add by beginning at the left, but it will be seen on trial to be much less convenient. We commence at the bottom of a column to add as a matter of custom; in practice it is sometimes more convenient to begin at the bottom and at other times at the top.

Were the scale any other than the decimal, the principle and method of adding would be the same. In addition of denominate numbers, where the scales are irregular, the same general principle is employed. We find the sum of a lower order of units, reduce this to the next higher order, etc. The difference in practice is that, with the decimal scale, the reduction is evident from the notation, while in the irregular scales we must divide to make the reduction. The general principle of thought in the two cases is, however, identical.

CHAPTER II.

SUBTRACTION.

THE fundamental analytical process of arithmetic is *Subtraction.* This process arises from the reversing of the fundamental synthetic process. The primary operation of arithmetic, as previously seen, is synthesis. Every synthesis implies a corresponding analysis; hence, the second operation of arithmetic, as a logical consequence, must be the opposite of the primary synthetic process. In the former case we united numbers to find a sum; here we separate numbers to find a difference. This general analytic process has received the name of Subtraction.

Definition.—Subtraction is the process of finding the *difference* between two numbers. The *difference* between two numbers is a number which added to the less will give a sum equal to the greater. The greater number is called the *Minuend;* the less number is called the *Subtrahend.* Subtraction may also be defined as the process of finding how much greater one number is than another; or, as the process of finding a number which, added to the smaller of two numbers, will equal the greater. The definition first presented is, however, preferred.

Cases.—Subtraction is philosophically divided into two general cases, like addition. The first case consists in finding the difference between two numbers, independent of the notation used to express them. The second case consists in finding the difference between numbers as expressed in written characters, and thus grows out of the use of the Arabic notation. The first is a case of pure arithmetic, independent of any notation; the latter is incidental to the notation adopted to express numbers. The former deals with small numbers, and the process being wholly in the mind may be called *Mental Subtraction;* the latter is employed in subtracting large numbers expressed with written characters, and may be called *Written Subtraction.* The former is an independent process complete in itself; the latter has its origin in the Arabic system of notation, and is dependent upon the former for its elementary differences. In the ordinary text-books, the second case is usually divided into two separate cases, depending upon the size of the terms in the minuend and subtrahend; but such division is designed to simplify the subject in instruction, and is, therefore, a practical rather than a logical division of the subject.

Principles.—The operations in subtraction depend upon some general laws called principles. The most important of the fundamental principles of subtraction are the following:

1. *Similar numbers only can be subtracted.* Thus, we cannot find the difference between 9 apples and 4 peaches, for if we take the difference between the numbers 9 and 4, which is 5, it will be neither 5 apples nor 5 peaches. Suppose, however, that we have 9 apples and peaches, consisting of 5 apples and 4 peaches; can we then subtract 4 peaches, and will not the remainder be 5 apples? Or suppose we have a collection of knives and forks consisting of half a dozen of each, which are sometimes spoken of as "12 knives and forks;" can we not take away 6 forks and leave remaining 6 knives? In reply, we remark that such a "taking away" is not what we mean by subtraction, which is defined as the process of finding the difference of two numbers.

It is also manifest, as in addition, that if we regard the dissimilar numbers as having the same generic name, they will then become similar and we can subtract them. Thus, 9 *apples* and 4 *peaches* may be regarded as 9 *objects* and 4 *objects*, the difference of which is 5 *objects*. So in subtracting the different orders of units in the Arabic scale, we cannot subtract them directly as different orders, but by reducing them to the same denomination, the subtraction is readily performed.

2. *The difference is a number similar to the minuend and subtrahend.* This is a necessary truth intuitively apprehended. Thus 4 *men* subtracted from 9 *men*, leaves 5 *men*, and not 5 *girls*, or 5 *women*. If we have a group consisting of 9 persons, 5 *men* and 4 *women*, and take away 4 women, there will remain 5 men ; hence we might infer that 4 women taken from 9 persons leaves 5 men ; but this is not a universal truth; neither, as stated above, is such a taking away, what we mean by subtraction.

3. *If the minuend and subtrahend be equally increased or diminished, the remainder will be the same.* This is included in the axiom that the difference between two numbers equals the difference between them when equally increased or diminished. The truth of such a proposition is seen to be necessary as soon as the proposition is clearly apprehended by the mind.

4. *The minuend equals the sum of the subtrahend and remainder; the subtrahend equals the difference between the minuend and remainder.* These two principles flow from the conception of subtraction, and the relation of the several terms to one another. Given a clear idea of the process of subtraction, and the relation of the three terms in the process, and these truths immediately follow.

Method.—The two cases of subtraction, as of addition, require distinct methods of treatment. In the former case we subtract directly as *wholes*, finding the difference by reversing the process of addition. In the latter case we subtract by *parts*, using the elementary differences to find the differences of the corresponding parts. An explanation of both cases will be presented.

CASE I. *To find the primary differences in arithmetic.*— The elementary differences are obtained by a reversion of the process of finding the elementary sums. This may be done in two distinct ways. First, we may find the difference between two numbers by *counting off* from the larger number as many units as are contained in the smaller number. Thus, if we wish to subtract *four* from *nine*, we may begin at *nine* and count backward *four* units: thus, *eight, seven, six, five;* and finding that we reach *five*, we know that *four* from *nine* leaves *five*. This is the reverse of the process by which we obtained the elementary sums in addition. In one case we *count on* for the sum; in the other we *count off* for the difference.

The other method consists in finding the elementary differences by deriving them by inference from the elementary sums. Thus, in finding the difference between *five* and *nine*, we may proceed as follows: since *four* added to *five* equals *nine, nine* diminished by *five*, equals *four*. This process, put in a formal manner, is as follows: The difference between two numbers is a number which, added to the less, will equal the greater; but, *four* added to *five*, the less, equals *nine*, the greater; hence, *four* is the difference between *nine* and *five*. In other words, we know that *five* from *nine* leaves *four*, because *four* added to *five* equals *nine*.

The difference between these two methods is radical. By the former method we derive the difference by direct intuition, as we obtained the sums in addition. We *see* that the difference is *five*. By the second method we *infer* that the difference is *five*, without directly seeing it. The latter is a process of reasoning, and will admit of being reduced to the form of a syllogism, as is shown above. The point made here is an important one, and will throw some light on the nature of the science of arithmetic, which, by the metaphysicians, has been somewhat imperfectly understood.

The second method is preferred in practice to the first, as we can make use of the elementary sums in finding the elementary differences. If the first method is used, it will be necessary to commit the elementary differences as well as the elementary sums. By making the differences depend upon the sums, this labor will be avoided.

CASE II. *To subtract numbers expressed by the Arabic scale of notation.* With large numbers we cannot subtract the one directly from the other as with small numbers; we therefore divide the labor, subtracting by *parts;* that is, we find the difference between the corresponding groups of each term. By this means the labor of subtracting is greatly facilitated, so that with large numbers, which it would be almost, if not quite impossible otherwise to subtract, the operation becomes simple and easy.

In the subtraction of numbers expressed in the Arabic scale of notation, two distinct cases arise; first, when the number of each group of the subtrahend does not exceed the corresponding number of the minuend; second, when the number of a group in the subtrahend exceeds the corresponding number in the minuend. In the first case we readily subtract each group in the subtrahend from the corresponding group in the minuend. In the second case a difficulty arises, for which we have two distinct methods of explanation, called respectively the *Method by Borrowing*, and the *Method by Adding Ten.*

To illustrate these methods, suppose it be required to subtract 526 from 874.

OPERATION.

$$\begin{array}{r} 874 \\ 526 \\ \hline 348 \end{array}$$

First Method.—Having the numbers written as in the margin, we commence at the right to subtract, and reason thus: we cannot take 6 units from 4 units, we will therefore take 1 ten from the 7 tens, and add it to the four units, which will give 14 units. We then subtract 6 units from 14 units, which gives 8 units. We then subtract 2 tens from the 6 tens which remain after taking away the 1 ten, which leaves 4 tens. We also subtract 5 hundreds from 8 hundreds, leaving 3 hundreds: hence the difference is 348.

Second Method.—By the second method we reason thus: We cannot subtract 6 units from 4 units, hence we add 10 to the 4, making 14 units, and then say, 6 units from 14 units leave 8 units. Now, since we have added 10 to the minuend, that the remainder may be correct we must add *one ten* to the subtrahend; hence we have 3 tens from 7 tens leave 4 tens, and also as before, 5 hundreds from 8 hundreds, 3 hundreds. This solution is founded upon the principle that *the difference between two numbers equals the difference between the two numbers equally increased.*

The first method seems preferable on account of its simplicity of thought, as it merely changes the form of the minuend. Pupils see the reason of the process by this method more readily than by the method of adding ten. The second method, however, is preferred by some teachers for at least two reasons. First, it is the method generally used in practice; nearly all persons increasing the next lower term after "borrowing," instead of diminishing the upper one. Second, it is, in many cases which arise, much more convenient than the other method, as in subtracting 12345 from 20000. By the second method, the solution of this problem will be much simpler than by the first.

Another Method.—There is still another method of subtracting, which, if not of any practical value, is at least of sufficient interest to be worthy of mention. It consists in subtracting the terms of the subtrahend from 10, and adding the difference to the corresponding terms of the minuend. Thus, in subtracting 27865 from 74682, we say 5 from 10 leaves 5, and 2 are 7; 6 and 1 to carry are 7, and 7 from 10 leaves 3, and 8 are 11; set down the 1; 8 from 10 leaves 2, and 6 are 8; 7 and 1 to carry are 8, and 8 from 10 leaves 2, and 4 are 6, etc.

$$\begin{array}{r} 74682 \\ \underline{27865} \\ 46817 \end{array}$$

Rule.—In the rule for subtraction, arithmeticians make the same mistake as in the rule for addition. Thus, they say, "Subtract each figure of the subtrahend from the figure above it in the minuend," or "take each figure of the subtrahend from the figure above it," or, "if a figure in the lower number is larger than the one above it," etc. These errors are almost inexcusable. We cannot subtract *figures*, we subtract *numbers*. If we "take one figure from another" the other figure will be left, not the difference of the numbers expressed by them. A figure is larger or smaller according to the kind of type in which it is printed. The figure *two* may be large (**2**) or small (2). One figure may be larger than another, and express a smaller number; as, **3** and 8.

This error may be avoided by the use of the word *term* for the number expressed by the figure. The rule will then read, "Begin at the right and take each term of the subtrahend from the corresponding term of the minuend," etc. "If a term of the subtrahend is greater than the corresponding term of the minuend," etc.

Remarks.—We write terms of the same order in the same vertical column for *convenience* in subtracting, since only numbers of the same group can be subtracted. We commence at the right, so that when a term of the subtrahend expresses more units than the corresponding term of the minuend, we may take it from the next higher group of the minuend; or, if we use the other method of subtracting, that we may add 10 of a group to the minuend, and 1 of the next higher group to the subtrahend; in other words, we commence at the right as a matter of *convenience*, as will be seen in the attempt to subtract by commencing at the left.

The taking *one* from the next term of the minuend is called "*borrowing*," and the adding *one* to the next term of the subtrahend is called "*carrying*." The accuracy of these words has been questioned. To *borrow* is to obtain that which we expect to return to the one from whom we borrow. It does not seem much like "borrowing" to take from one thing and return what we take to another. It is something like "robbing Peter to pay Paul." In regard to the term "carrying," it may be asked in what it is carried; though we may answer, as the boy did, "we carry in the head." Notwithstanding these objections, the terms *borrowing* and *carrying* have been sanctioned by good usage; and, since custom is the lawgiver in language, we may accept them as correct. Their use is a matter of convenience, also, as they indicate operations for which we have no other technical terms. It may be remarked that it required many years for the people of Europe to become familiar with the processes of *borrowing* and *carrying*. In a work on arithmetic by Bernard Lamy, published at Amsterdam in 1692, the author states that a friend sends him the mode of using the *carriage* in subtraction, he having previously borrowed from the upper line; and this is presented as a novelty.

CHAPTER III.

MULTIPLICATION.

THE general process of synthesis is *Addition*. Having become familiar with this general synthetic process in accordance with the law of thought, *from the universal to the particular*, we begin to impose certain conditions upon it. The numbers primarily united were of any relative value; if, now, we impose the condition that the numbers united shall be all *equal*, with the new idea of the *times* the number is used, we have a new process of synthesis, which we call *Multiplication*.

Multiplication is thus seen to be a special case of addition, in which the numbers added are all equal. The idea of multiplication is contained in addition, and is an outgrowth of it. They are both synthetic processes—one being a general, and the other a more special synthesis. Multiplication, however, involves the idea of "*times*," which does not appear in addition. This notion of "*times*," originating in multiplication, is one of the most important in mathematics, and is itself the source of a large portion of the science. Thus, in involution there is no apparent trace of the idea of addition, and the same is true in respect of other processes. If, however, we follow these processes back far enough, we shall find they have their origin in the primary process of addition. Even involution may be performed by successive additions.

Definition.—Multiplication is the process of finding the *product* of two numbers. The *Product* of two numbers is the result obtained by taking one number as many times as there are units in the other. The number multiplied is called the *Multiplicand.* The number by which we multiply is called the *Multiplier.*

This definition of multiplication, introducing the word *Product,* makes it similar to the definitions of addition and subtraction, in which the terms *sum* and *difference* are used. Defining Division in a similar manner by using the word *Quotient,* we shall have a harmony in the definitions of the four fundamental rules, which has not hitherto existed. I have adopted this method in my Higher Arithmetic, and shall introduce it into my other mathematical works.

Multiplication is usually defined as the process of taking one number as many times as there are units in another. This definition is not entirely satisfactory. It says nothing about finding a *result,* which is specified in the definitions of addition and subtraction, and which seems to be necessary also here. To supply this omission, I have previously defined multiplication as the process of finding the result of taking one number as many times as there are units in another. After a very careful consideration of the subject, however, I have concluded to adopt the method of defining multiplication as *the process of finding the product,* thus securing a uniformity in the definitions of the fundamental operations.

Principles.—The operations of multiplication are founded upon certain necessary truths called principles. The most important of the principles of multiplication are those which follow:

1. *The multiplier is always an abstract number.* For, the multiplier shows the *number of times* the multiplicand is taken, and hence must be abstract, since we cannot take anything *yards times* or *bushels times,* etc. From this it follows that such problems as "Multiply 25 cts. by 25 cts.," or "2s. 6d. by itself" are impossible and absurd. In finding areas and volumes, we speak of multiplying feet by feet for square feet,

square feet by feet for cubic feet, etc. It should be remembered, however, that this is merely a convenient expression, which does not indicate the actual process. In finding the area of a rectangle, we multiply the number of square feet on the base by the number of such rows; the multiplicand being square feet and the multiplier an abstract number.

2. *The product is always similar to the multiplicand.* This is manifest from the fact that the product is merely the sum of the multiplicand used as many times as there are units in the multiplier. Thus, 3 times 4 *apples* are 12 *apples*, and cannot be 12 *pears* or *peaches*.

3. *The product of two numbers is the same, whichever is made the multiplier.* This may be seen by placing 3 rows of 4 stars each in the form of a rectangle, as in the margin. Now these may be regarded as 3 rows of 4 stars each, or 4 rows of 3 stars each; hence 3 times 4 is the same as 4 times 3; and the same may be shown for any other two numbers.

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4. *If the multiplicand be multiplied by all the parts of the multiplier, the sum of all the partial products will be the true product.* This grows out of the general principle that the whole is equal to the combination of all of its parts. It is applied in finding the product of two numbers expressed by the Arabic system.

5. *The multiplicand equals the quotient of the product divided by the multiplier; the multiplier equals the quotient of the product divided by the multiplicand.* These two principles are manifest to the mind as soon as it attains a clear idea of the processes of multiplication and division, and the relation of the two to each other.

Cases.—Multiplication is philosophically divided into two general cases. The first case consists in finding the products of numbers independently of the method of notation used to express them. The second case is that which grows out of the use of the Arabic system of notation. The former deals with small numbers mentally, and may be called *Mental Multiplication;* the latter deals with large numbers, expressed by means of written characters, and may be called *Written Multiplication.* The former is an independent process complete in itself, and belongs to pure number; the latter has its origin in the Arabic system, and is dependent upon the former for its elementary products.

Method.—The general method is to find the product of small numbers by addition, and then use these in the multiplication of large numbers. The first case is thus made to depend upon addition, and the second case upon the first case. Both cases will be formally presented.

CASE I. *To find the elementary products of arithmetic.* The first object in multiplication is to find the *elementary products.* By the elementary products are meant the products of small numbers which, arranged together, constitute what is called the *Multiplication Table.* These elementary products are derived by addition. Thus, we ascertain that *four* times *five* are *twenty,* by finding, by actual addition, that the sum of *four fives* is *twenty.* In this manner all the elementary products of the table were originally obtained. This table is committed to memory in order to save labor and facilitate the process of calculation. We are thus able to tell immediately the product of two small numbers, which otherwise we should be obliged to obtain by an actual addition.

The elementary products are not derived by intuition, and are therefore not axioms; they are the result of a process of reasoning. Thus, in order to find the product of *three times four,* we may reason as follows: *Three times four* is equal to the sum of *three fours;* but the sum of *three fours,* we find by addition, is *twelve;* hence, *three times four* is *twelve.* This is as valid a syllogism as "A is equal to B; but B is equal to C; hence, A is equal to C."

The extent of the table, for all practical purposes, is limited by "nine times nine." That is, with our Arabic system of notation and the decimal method of numeration, it is not necessary that the elementary products should extend beyond "nine times." It is not at all inconvenient, however, but quite natural that it should include *eleven* and *twelve* times, since the names *eleven* and *twelve* are a seeming departure from the decimal system of numeration.

Case II. *To multiply numbers expressed by the Arabic system of notation.* When the numbers are small, as we have seen, we multiply them directly as wholes; when we extend beyond the elementary products, the principle is to multiply *by parts*. Thus, instead of multiplying the multiplicand as a single number, we multiply first one group, then the next group, and so on, as we united numbers in addition. Also, when the multiplier exceeds *nine*—or in practice, *twelve*—that is, when it is expressed in two or more places, we multiply first by the units term, then by the tens term, etc.; and then take the sum of these partial products.

To illustrate, let it be required to multiply 65 by 37. To multiply by *thirty-seven* as a single number, would be quite a difficult task. We do not attempt this, however, but first multiply by 7 units, one part of 37, and then by 3 tens, the other part of 37, and then take the sum of these products. It is also seen that the number 65 is not multiplied as a single number, but by using its parts, 5 units and 6 tens. The method of explaining the process is as follows:

OPERATION.

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  65
  37
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 455
195
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2405
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Solution.—Thirty-seven times 65 equals 7 times 65 plus 3 tens times 65. Seven times 5 units are 35 units, or 3 tens and 5 units; we write the 5 units, and reserve the 3 tens to add to the product of tens. Seven times 6 tens are 42 tens, which, increased by 3 tens, equals 45 tens, or 5 tens and 4 hundreds, which we write in its proper place. Multiplying similarly by 3 tens, we have 5 tens 9 hundreds and 1 thousand; and taking the sum of these two partial products, we have 2405.

This method of multiplication is founded upon, and is only possible with a system of notation similar to the Arabic. Without some such method of expressing numbers in characters, the multiplication of large numbers would be exceedingly laborious, if not altogether impossible. We are thus continually reminded of the advantages of the Arabic system of notation, and learn almost to venerate the people and country that conferred so great a boon upon the human race by its invention.

Rule.—The error of confounding the meaning of figure and number is repeated in the rule for multiplication. The rule, as usually given is, "Multiply each figure of the multiplicand by the multiplier," etc., or "Multiply the multiplicand by each figure of the multiplier," etc. This error is easily avoided by the use of the word *term* for figure. It should be remembered that we have two distinct things, the *number* and the *numerical expression*. The parts of the numerical expression are *figures*; the parts of the entire number are *numbers*. The word *term* may be employed to express both of these, without any obscurity and with much convenience. The rule will then read, "Multiply each term of the multiplicand by the multiplier," etc., or, "by each term of the multiplier," etc.

Remark.—We write the numbers as indicated above for convenience in multiplying. The placing of the multiplier under the multiplicand, instead of over it, and multiplying from below, is a mere matter of custom, corresponding with the method of adding and subtracting. We begin at the right hand to multiply so that when any product exceeds *nine*, we may incorporate the number expressed by the left hand figure with the following product. The convenience of this will be readily appreciated by performing the multiplication by beginning at the left. It was formerly the custom, however, to begin at the left, writing the partial products in their order and subsequently collecting them.

CHAPTER IV

DIVISION.

THE general process of analysis is Subtraction. After the mind becomes familiar with this general process, it begins to extend and specialize it, and thus arises a new process called *Division*. Division is, therefore, a special case of subtraction, in which the same number is to be successively subtracted with the object of finding how many times it is contained. The idea of Division is thus seen to be contained in that of Subtraction, and is the outgrowth of it.

Division may also be regarded as arising from a reversing of the process of multiplication. In multiplication, we obtain the product of two numbers; and since the product is a number of times the multiplicand, we may regard it as containing the multiplicand a number of times. Thus, since *four times five* are *twenty*, *twenty* may be considered as containing *five, four times*. Division is thus regarded as an analytic process, arising from reversing the synthetic process of multiplication.

It thus appears that Division may have originated in either of two different ways. In which way it did actually arise, it is impossible for us to decide with certainty. It has generally been supposed, judging from the old definition that "Division is a concise method of Subtraction," that it had its genesis in Subtraction. My own opinion, however, is that it originated by reversing multiplication, for which I state the following reasons:—First, as subtraction arose from reversing the process of addition, so is it natural to suppose that division, a concise subtraction, would arise from reversing multiplication, a concise addition. Second, division involves, as essential to it, the idea of "*times*," which had already appeared in multiplication. It seems much more natural to take the idea of *times* from multiplication, where it already existed, than to originate it from the process of subtraction.

Definition.—Division is the process of finding the *quotient* of two numbers. The *quotient* of two numbers is the number of times that one number contains the other. The number divided is the *Dividend;* the number we divide by is the *Divisor.* The definition usually given is, "Division is the process of finding how many times one number is contained in another." This is regarded as correct, but is less simple and concise than the one above suggested.

Defining division in this manner, we have a simple and concise definition, easily understood and logically accurate. It follows the method generally adopted for addition and subtraction, and which I have also suggested for multiplication; and presents a happy uniformity in the definitions of the four fundamental operations of arithmetic. The objects of these four fundamental processes, as thus presented, will respectively be to find the *Sum*, the *Difference*, the *Product*, and the *Quotient* of numbers.

Principles.—The operations in division are controlled by certain necessary laws of thought to which we give the name of principles. The following are the most important of the principles of division:

1. *The dividend and divisor are always similar numbers.* This is true of division scientifically considered, as may be seen by regarding it as originating in subtraction or multiplication. Supposing that it has its root in subtraction, and remembering that in subtraction the two terms must be alike, we see that this principle follows of necessity. Thus, if we inquire how many times one number is contained in another,

it is evident that these numbers must be similar. We may inquire how many times 4 apples are contained in 8 apples, but not how many times 4 peaches are contained in 8 apples. Neither can we say "How many times is 4 contained in 8 apples?" for 8 apples will not contain the abstract number 4 any number of times. The same conclusion is reached if we regard division as originating in multiplication. If we assume that 4 is contained in 8 apples 2 apples times, it would follow that 2 apples times 4 equals 8 apples, which is absurd.

Several recent writers take the position that a concrete number may be divided by an abstract number, because in practice we thus divide a concrete number into equal parts. This is a subordination of science to practice, which is neither philosophical nor necessary. The practical case which they thus try to include in the theory of the subject, admits of a scientific and simple explanation, without any modification of the fundamental idea of division; and when thus explained it becomes apparent that the two terms are similar numbers.

2. *The quotient is always an abstract number.* This results from the fundamental idea of division, whether we regard it as originating in subtraction or multiplication. The quotient shows *how many times* one number is contained in another, and one number cannot be contained in another number *yards times*, or *apples times*, etc., from which it follows that the quotient must be abstract. The quotient shows how many times one number may be subtracted from or taken out of another before exhausting the latter, and must therefore be a number of times, and consequently abstract. Or, regarding it as arising from multiplication, the quotient is the number of times the divisor which equals the dividend; and, as such, is a *multiplier;* and must, consequently, be abstract. Suppose it were said that 2 is contained in 8 apples, "4 apples times,"—and all authors agree as to the quotient denoting the number of times the divisor is contained in the dividend—then it would follow that "4 apples times" 2 are 8 apples; which is, of course, absurd.

3. *The remainder is always similar to the dividend.* This is evident, since the remainder is an undivided part of the dividend. In practice, as above intimated, some of these principles seem to be violated, but if the analysis be given, it will be seen that the violation is merely seeming, and not actual.

4. The following principles show the relation of the terms in division:

1. *The dividend equals the product of the divisor and quotient.*

2. *The divisor equals the quotient of the dividend and quotient.*

3. *The dividend equals the product of the divisor and quotient, plus the remainder.*

4. *The divisor equals the dividend minus the remainder, divided by the quotient.*

5. The following principles show the result of multiplying or dividing the terms in division:

1. *Multiplying the dividend or dividing the divisor by any number multiplies the quotient by that number.*

2. *Dividing the dividend or multiplying the divisor by any number divides the quotient by that number.*

3. *Multiplying or dividing both divisor and dividend by the same number does not change the quotient.*

Cases.—Division is philosophically divided into two general cases. The first case consists in finding the quotient of numbers independently of the method of notation used to express them. The second case is that which grows out of the use of the Arabic system of notation. The former case deals with small numbers mentally, and may be called *Mental Division;* the latter deals with large numbers, expressed by means of written characters, and may be called *Written Division.* The former is an independent process, belonging to pure number, and is complete in itself; the latter operates by means of the Arabic characters, and is dependent upon the former for its elementary quotients.

Method.—In division we first find the elementary quotients corresponding to the elementary products of the multiplication table. These may be obtained in two different ways, as will be explained. In the second case we operate by parts, using the elementary quotients as a basis of operation. The two cases will be formally presented.

CASE I. *To find the elementary quotients of arithmetic.* The first object in division is to find the elementary quotients corresponding to the elementary products of the multiplication table. These quotients admit of a double origin; that is, they may be derived by the method of *concise subtraction*, or of *reverse multiplication.* Thus, if we wish to ascertain how many times *five* is contained in *twenty*, we may find how many times *five* can be taken out of *twenty* by subtraction, and this will show how many times *twenty* contains *five.* This is the method of subtraction, and as thus viewed, division may be regarded as a method of concise subtraction. Again, since we know that *four times five* are *twenty*, we can immediately infer that *twenty* contains *four fives*, or that *twenty* contains *five four* times. This is the method of multiplication, and as thus viewed, division may be regarded as a method of reverse multiplication.

Either of these two methods may be used for finding the elementary quotients, but the method of *reverse multiplication* is much more convenient in practice. The quotients are immediately derived from the products of the multiplication table, and we are thus saved the labor of forming and committing a table of division. If, however, the elementary quotients be derived by subtraction, it will be necessary to construct a division table, and commit the quotients, as we do the products in multiplication.

These elementary quotients, whether derived by multiplication or subtraction, are the result of a process of reasoning. The process of thought may be illustrated in the problem, "*Five* is contained how many times in *twenty?*" and is as follows:

Five is contained as many times in *twenty* as *twenty* is *times* five; but twenty is *four times* five; hence, *five* is contained in *twenty, four times.* In ordinary language, this is abbreviated thus: five is contained *four times* in *twenty,* since *four times* five are *twenty.*

By the method of subtraction we reason thus: *five* is contained as many times in *twenty* as *five* can be successively subtracted from or taken out of *twenty;* but *five* can be successively subtracted from *twenty, four times;* hence, *five* is contained *four times* in twenty. The ordinary form of thought is, *five* is contained *four times* in *twenty,* since it can be subtracted from *twenty, four times.* By "subtracted from," as here used, we mean subtracted successively from until twenty is exhausted.

CASE II. *To divide when the numbers are expressed in the Arabic scale of notation.* When the numbers are small, we divide them, as we have seen, directly as wholes; when we extend beyond the elementary quotients, the principle is to divide *by parts.* The dividend is not immediately divided as a whole, but is regarded as consisting of parts or groups; and these are so divided that, when remainders occur, they may be incorporated with inferior groups, and thus the whole number be divided. This method, as in multiplication, is due to the system of Arabic notation, and enables us to divide large numbers, which would be exceedingly difficult, if not impossible, with a different system of notation.

In *Written Division,* or division of large numbers, two cases are presented. First, when the divisor is so small that only the *elementary dividends* and *divisors* are used; second, when the divisors and dividends are larger than those employed in obtaining the elementary quotients. The methods of treating these two cases are distinguished as *Short Division* and *Long Division.* In *Short Division,* the partial dividends are not written; in *Long Division,* the partial dividends and other necessary work are written.

Illustration.—To illustrate the method of *Short Division*, divide 537 by 3. Here we cannot divide the given number as *a whole*, that is, as *five hundred and thirty-seven*, but by separating it into *parts*, we can readily divide these parts, as they give only the elementary quotients. Thus, we first divide *five* hundred, reduce the remainder of the group to tens and incorporate with the tens group, making 23 tens, divide this as before, and thus continue until the whole of the number has been divided.

When the divisor is greater than 12, the division can no longer be performed by using the elementary dividends and quotients. The process then becomes more difficult, although it involves the same principles as when smaller numbers are used. As the elementary quotients were derived from multiplication, so in *Long Division* we determine the quotient by multiplying. We multiply the divisor by some number which we suppose to be the quotient term, and if the product does not exceed the partial dividend, nor the difference between the product and partial dividend exceed the divisor, we know that we have obtained the correct quotient figure. The method described is so common that it need not be illustrated by a problem.

Rule.—The mistake of using *figure* for *number* is also made in stating the rule for division. One author says, "Find how many times the divisor is contained in the fewest figures on the left of the dividend," etc.; another says, "Take for the first partial dividend the fewest figures of the given dividend," etc.; another says, "Take for the first partial dividend the least number of figures on the left that will contain the divisor," etc. Of course, figures will not contain the divisor; *the number expressed by the figures* is what is intended, and therefore should be expressed. The error may be corrected by saying, "Divide the number expressed by the fewest figures on the left that will contain the divisor," or, "by the fewest terms," etc

Remark.—We write the divisor at the left of the dividend and the quotient at the right as a matter of custom. Some prefer writing the divisor at the right and placing the quotient under the divisor. We begin at the left to divide, so that the remainder, when one occurs, may be united with the number of units of the next lower order, giving a new partial dividend. If we attempt to divide by beginning at the right, we will see the advantage of the ordinary method.

SECTION II.

DERIVATIVE OPERATIONS OF SYNTHESIS AND ANALYSIS.

I. Introduction.

II. Composition.

III. Factoring.

IV. Common Divisor.

V. Common Multiple.

VI. Involution.

VII. Evolution.

CHAPTER I.

INTRODUCTION TO DERIVATIVE OPERATIONS.

THE four Fundamental Operations are the direct and immediate outgrowth of the general processes of synthesis and analysis as applied to numbers. They are called Fundamental Operations because all the other operations involve one or more of these, and may be regarded as being based upon them. They are the foundation or basis upon which the others are built up, the germ from which they are evolved, the soil out of which they grow.

Several of the processes of arithmetic are so intimately related to the fundamental operations that they may be regarded as directly originating in and growing out of them. Such are the processes of *Factoring*, *Common Multiple*, *Common Divisor*, etc. These processes have their roots in the general notions of the fundamental operations, and are evolved from them by a modification and extension of the primary analytic and synthetic processes. They are developed by the thought process of comparison, though they have not their basis in comparison, like the processes of *Ratio*, *Proportion*, etc. Being thus derived from the fundamental operations, they may be called the *Derivative Operations* of synthesis and analysis. Let us notice the origin and nature of these derivative operations.

If two or more numbers are multiplied together, and the result is considered with respect to its elements, we have the idea of a *Composite Number*. The general process of forming composite numbers may be called *Composition*. The numbers synthetized in forming a composite number are called *Factors* of that number. If we form a composite number consisting of *two equal* factors, we have a *square;* of *three equal* factors, a *cube*, etc., and the process is called *Involution.* If we find a composite number which is a number of times each of several numbers, or is so composed that each of them is one of its factors, it is called a *common multiple* of these numbers, and the process is known as finding *Common Multiples.*

These processes are distinct from Multiplication, though related to it. They employ multiplication and are the outgrowth of the general multiplicative idea, but pass beyond the primary idea of multiplication. In multiplication, the main idea is the *operation* of repeating one number as many times as there are units in another to obtain a result; here the thought is the *result* of the operation *compared* with the numbers multiplied together. In the former case, the process is purely synthetic; here comparison unites with synthesis, and employs it for a particular object. The operation of multiplying is assumed as a fact, and employed for the purpose of attaining a result bearing some relation to the elements combined.

Having obtained composite numbers, and the idea of their being composed of factors, we naturally begin to analyze them into their elements in order to discover these factors. This gives rise to an analytic process, the converse of *Composition.* The general process of analyzing a number into its factors is called *Factoring.* If we resolve a number into several *equal factors* for the purpose of seeing what factor must be repeated two, or three, etc., times to produce the number, we have a process known as *Evolution.* If we have given several numbers, and proceed to find a *common factor* of these numbers, we have the process known as *Common Divisor.*

These processes, though related to Division, are clearly distinguished from it. They are an outgrowth of the general idea of division, but extend beyond it. In division it is the operation of finding how many times one number is contained in another that is the prominent idea; here the idea is the result considered in relation to the number or numbers operated upon. In *Factoring*, the process of comparison enters as an important element. Division is a process purely analytical; Factoring is analysis, and more; it is *analysis plus comparison*. It has its root in Analysis, and is developed by the thought-process of Comparison.

There are, therefore, two general derivative processes, *Composition* and *Factoring*, each of which embraces corresponding and opposite processes. The terms, *Composition* and *Factoring*, are in practice restricted to the general processes; the special processes are known by their particular names. We have thus three pairs of derivative processes,—*Composition* and *Factoring*, *Multiples* and *Divisors*, and *Involution* and *Evolution*. These will be treated in successive chapters.

CHAPTER II.

COMPOSITION.

COMPOSITION is the process of forming composite numbers when their factors are given. It is a general process which contains several subordinate and special ones. When fully analyzed, it will be seen to present several interesting cases besides the more particular ones of Involution and Multiples. From the previous analysis it is seen that there is a real case of Synthesis, the converse of the analytic process of Factoring.

This new generalization, and the term I have applied to it, will, I trust, receive the approval of mathematicians. Its importance as a logical necessity, is seen in its relation to Factoring. In the fundamental operations each synthetic process has its corresponding analytic process. Thus, addition is synthetic, subtraction is analytic; multiplication is synthetic, division is analytic. It follows, therefore, that there should be a synthetic process corresponding to the analytic process of *Factoring.* This process I have presented under the name of *Composition*, or the process of forming composite numbers.

Cases.—There are several interesting and practical cases of Composition, some of the most important of which are the following:

I. To form a composite number out of any factors.

II. To form a composite number out of equal factors.

III. To form a composite number out of factors bearing any definite relation to each other.

IV. To form composite numbers which have one or more given common factors.

V. To form several or all of the composite numbers possible out of given factors.

VI. To determine the number of composite numbers that can be formed out of given factors.

Method of Treatment.—The method of treatment is to combine these factors by multiplication in such a manner as to attain the result desired. I will briefly state the manner of treating each case.

CASE I. *To form a composite number out of any factors.* In Case I. we find the result by simply taking the product of the factors. Thus the composite number formed from the factors 2, 3, and 4 equals $2\times3\times4$, or 24.

CASE II. *To form a composite number out of equal factors.* Case II. may be solved in the same manner as Case I., or we may multiply a partial result by itself or by another partial result, to obtain the entire result. Thus, if we wish to find the composite number consisting of eight 2's, we may multiply 2 by 2, giving 4, then multiply 4 by 4, giving 16, and then multiply 16 by 16, giving 256, the number required.

CASE III. *To form a composite number out of factors bearing any definite relation to each other.* In this case we may have given one factor and the relation of the other factors to it; we first find the factors and then take their product. Thus, required the number consisting of three factors, the first being 4, the second twice the first, and the third three times the second. Here, we first find the second factor to be 8, and the third to be 24, and then take the product of 4, 8, and 24, which we find to be 768.

CASE IV. *To form composite numbers which have one or more given common factors.* This case may be solved by taking the given common factor, and multiplying it by any other factors we choose. If it is required that the factor given be the largest common factor of the numbers obtained, the multipliers selected must be prime to each other. To illustrate, find three numbers whose largest common factor shall be 12. If we multiply 12 by 2, 4, and 6, we will have 24, 48, and 72, three numbers whose common factor is 12; but since the numbers used as multipliers have a common factor, 12 is not the largest factor common to these three numbers. To find three numbers having 12 as their largest common factor, we may multiply 12 by 2, 3, and 5, which gives us the numbers 24, 36, and 60, in which 12 is the largest common factor.

Case V. *To form several or all of the composite numbers possible out of given factors.* In this case we may take the factors two together, three together, etc., until they are taken all together; or we may multiply 1 and the first factor by 1 and the second factor, the products thus obtained by 1 and the third factor, etc., until all the factors are used. To illustrate, form all the possible composite numbers out of 2, 3, 5, and 7.

We first find all the possible products taking them two together; then all the products taking them three together, and then the products taking them four together, as is shown in the margin. Another method, not quite so simple in thought but more convenient in practice, is as follows:

OPERATION.

$2\times3=6 \quad 3\times5=15$
$2\times5=10 \quad 3\times7=21$
$2\times7=14 \quad 5\times7=35$
$2\times3\times5=30$
$2\times3\times7=42$
$2\times5\times7=70$
$3\times5\times7=105$
$2\times3\times5\times7=210$

Multiplying 1 and 2 by 1 and 3, will give 1, 2, 3, and all the composite numbers that can be formed out of 2 and 3; these multiplied by 1 and 5 will give 1, 2, 3, 5, and all the composite numbers that can be formed out of 2, 3, and 5; these multiplied by 1 and 7 will give 1, 2, 3, 5,

OPERATION.

1 2
1 3
____
1 2 3 6
1 5
____
1 2 3 5 6 10 15 30
1 7
____
1 2 3 5 6 10 15 30 7 14 21 42 35 70 105 210

7, and all the composite numbers that can be formed out of 2, 3, 5, and 7. Omitting 1, 2, 3, 5, and 7 in the last result, and we have all the composite numbers that can be formed out of 2, 3, 5, and 7.

If some of the given factors are alike, we have an interesting modification of this case. Thus, suppose we wish to find the composite numbers which can be composed out of 2, 2, 2, 3, and 3. In this problem since 2 is used three times we may make the first series $1, 2, 2^2$, and 2^3, or 1, 2, 4, and 8; and since 3 is used twice, the second series will be 1, 3, and 3^2, or 1, 3, and 9; and the products of these, omitting 1, 2, and 3, will be the composite numbers required.

OPERATION.

1	2	4	8								
1	3	9									
1	2	3	4	6	8	9	12	18	24	36	72

CASE VI. *To determine the number of composite numbers that can be formed out of given factors.* We may solve this case by increasing the number of times each factor is used by unity, take the product of the results and diminish it by the number of different factors used increased by one. The reason for this method may be readily shown. Suppose we wish to find how many composite numbers can be formed with three 2's and two 3's.

Here we see that 2 used three times as a factor gives with 1 a series of *four* terms; and 3 used twice as a factor gives with 1 a series of *three* terms; hence the product will give a series of 4×3 or 12 terms, and omitting the unit and 2 and 3, we have *nine* terms. The inference from this solution will give the method stated above.

CHAPTER III.

FACTORING.

FACTORING is the process of finding the factors of composite numbers. It is the reverse of Composition. In Composition we have given the factors to find the number; in Factoring we have given the number to find the factors. Composition is a synthetic process; it proceeds from the parts by multiplication to the whole. Factoring is an analytic process; it proceeds from the whole by division to the parts.

A *Factor*, as now generally presented in arithmetic, is regarded as a *divisor* of a number, rather than a *maker* or producer of the number. This I regard as an error. The origin of the word, *facio*, I make, indicates its original meaning to be a maker of a composite number. The fact of a Factor of a number being a divisor of it is a derivative idea, resulting from the primary conception of its entering into the composition of the number. This primary idea of the office of a Factor is the one that should be primarily presented to pupils, rather than the secondary or derivative idea. We should define according to the fundamental, rather than the derivative office. To do otherwise is to invert the logical relation of ideas, and must, as I have known it, tend to confusion. Thus taught, it is seen that the proposition, *a factor of a number is a divisor of the number*, is an immediate inference, which would have to be inverted if the secondary office of a factor is made the fundamental idea.

Cases.—Factoring presents several cases analogous to those of Composition. Some of the principal ones are the following, which, it will be noticed, are the correlatives of those given under Composition.

I. To resolve a number into its prime factors.

II. To resolve a number into equal factors.

III. To resolve a number into factors bearing a certain relation to each other.

IV. To find the divisors common to two or more numbers.

V. To find all the factors or divisors of a number.

VI. To find the number of divisors of a number.

Method.—The general method of treatment is to resolve the number or numbers into their prime factors, and then combine these factors when necessary so as to give the required result. The prime factors of a number are found by division, and consequently it is convenient to know before trial what numbers are composite and can be factored, and the conditions of their divisibility. Hence, the subject of Factoring gives rise to the investigation of the methods of determining prime and composite numbers, and the conditions of the divisibility of composite numbers. This subject will be treated under the head of Prime and Composite Numbers. The method of treating each of the above named cases of factoring will be briefly stated.

CASE I. *To resolve a number into its prime factors.* In Case I. we divide the number by any prime number greater than 1 which will exactly divide it; divide the quotient, if composite, in the same manner; and thus continue until the quotient is prime. The divisors and the last quotient will be the prime factors required.

Thus, suppose we have given 105 to find its prime factors. Dividing 105 by the prime factor 3, and the quotient 35 by 5, we see that 105 is composed of the three factors 3, 5, and 7, and since these are prime numbers, its prime factors are 3, 5, and 7.

```
3)105
5)35
   7
```

CASE II. *To resolve a number into equal factors.* In Case II. we resolve the number into its prime factors and then combine by multiplication *one* from each set of *two equal* factors, when we wish *one* of the *two* equal factors of the number; *one* from each set of *three equal* factors when we wish *one* of *three* equal factors, etc.

Thus, suppose we wish to find the three equal factors of 216, or one of its three equal factors. We first resolve 216 into its prime factors, finding $216=2\times2\times2\times3\times3\times3$. Since there are *three* 2's, one of the three equal factors will contain 2; and since there are *three* 3's, one of the three equal factors will contain 3; hence one of the three equal factors is 2×3, or 6.

$$216=\begin{cases}2\times2\times2\times\\3\times3\times3\end{cases}$$
$$2\times3=6$$

CASE III. *To resolve a number into factors bearing a certain relation to each other.* In this case we may divide the given number by the product of the numbers representing the relation of the other factors to the smallest factor, then resolve the quotient into equal factors, and then multiply this equal factor by the numbers indicating the relation of the other factors to it.

Thus, resolve 384 into three factors, such that the second shall be twice the first and the third three times the first. Since the second factor equals 2 times the first and the third equals 3 times the first, the product of the factors will equal 2×3, or 6 times the first factor, used three times; hence if we divide 384 by 6, the quotient, 64, will be the product of the smallest factor used three times; therefore, if we resolve 64 into three equal factors, one of these factors will be the smallest of the three factors required. One of the three equal factors of 64, found by the previous case, is 4; hence, the smallest factor is 4, the second is 4×2 or 8, and the third is 4×3 or 12.

$$6)\underline{384}$$
$$64=4\times4\times4$$
$$2\times4=8$$
$$3\times4=12$$

CASE IV. *To find the divisors common to two or more* *numbers* In this case we resolve the numbers into their prime factors, and the common prime factors and all the numbers which we can form by combining them will be all the common divisors.

Thus, find the divisors common to 108 and 144. Resolving the numbers into their prime factors, we find the common factors to be $2^2 \times 3^2$; hence, 1, 2, 4, 3, 9, and all the possible products arising from their combination, will be all the divisors of 108 and 144.

OPERATION.

$$108 = 2^2 \times 3^3$$
$$144 = 2^4 \times 3^2$$
$$\text{Com. factor} = 2^2 \times 3^2$$

1 3 9
1 2 4
1 3 9 2 6 18 4 12 36

CASE V. *To find all the factors or divisors of a number* In this case we resolve the number into its prime factors, form a series consisting of 1 and the successive powers of one factor, and under this write 1 and the successive powers of another factor, and take the products of the terms of this series, etc. Thus, find all the different divisors of 108.

The factors of 108 are two 2's and three 3's. Since 3 is a factor 3 times, 1, 3, 3^2, 3^3, is the first series of divisors; and since 2 is a factor twice, 1, 2, 2^2 is the second series of divisors; and the products of the terms of these two series will give the prime factors and all possible products of them; and therefore, all the divisors of the number.

OPERATION.

$$108 = 2 \times 2 \times 3 \times 3 \times 3$$

1 3 9 27
1 2 4
1 3 9 27 2 6 18 54 4 12 36 108

CASE VI. *To find the number of divisors of a number.* In this case we resolve the number into its prime factors, increase the number of times each factor is used by 1, and take the product of the results. Thus, find the number of divisors of 108.

OPERATION.

$$108=2^2\times 3^3$$

$$(2+1)\times(3+1)=12$$

Factoring, we find 108 equals $2^2\times 3^3$. Now it is evident that 1 with the first and second powers of 2 will give a series of *three* divisors; and 1 with the first, second and third powers of 3, will give a series of *four* divisors; hence their products will give a series of *three* times *four*, or 12 divisors.

CHAPTER IV.

THE GREATEST COMMON DIVISOR.

A DIVISOR of a number is a number which will exactly divide it. A number is said to exactly divide another when it is contained in it a whole number of times without a remainder. A *Common Divisor* of two or more numbers is a divisor common to all of them. The *Greatest Common Divisor* of several numbers is the greatest divisor common to all of them. By using the word *factor* to denote an exact integral divisor, we may define as follows:

A *Divisor* of a number is a factor of the number. A *Common Divisor* of two or more numbers is a factor common to all of them. The *Greatest Common Divisor* of several numbers is the greatest factor common to all of them. These definitions employ the term *factor* with a derivative signification. A factor is primarily one of the *makers* of a number, entering into its composition multiplicatively. From this it follows, however, that a factor is an integral divisor of a number, and as such, it may be conveniently and legitimately used in defining a common divisor.

In the subject of greatest common divisor, the term "divisor" is used in a sense somewhat special. It signifies an exact divisor—a number which is contained a whole number of times without a remainder. The word *measure* was formerly used instead of divisor, and is in some respects preferable to divisor. A common divisor of several numbers is appropriately called their common measure, since it is a common unit of measure of those numbers. The term measure, in this sense, originated in Geometry, where a line, surface, or volume which is contained in a given line, surface, or volume, is called the unit of measure of the quantity. In arithmetic, the term divisor is generally preferred.

Cases.—There are two general cases of greatest common divisor, growing out of a difference in the method of treatment adapted to the problems. When numbers are readily factored, we employ one method of operation; when they are not readily factored, we are obliged to employ another method. This dual division of the subject into two cases is thus seen to be founded, not upon any distinctions in the idea of the subject, but upon the method of operation adapted to the numbers given. These two cases are formally stated as follows:

I. To find the greatest common divisor when the numbers are readily factored.

II. To find the greatest common divisor when the numbers are not readily factored.

Treatment.—The general method of treatment in the first case is to analyze the numbers into their factors, and take the product of the common factors. In the second case the numbers are operated upon in such a manner as to remove all the factors not common, and thus cause the greatest common divisor to appear. These two methods will be made clear by their application.

Case I. *To find the greatest common divisor when the numbers are readily factored.*

This case may be solved by two distinct methods. The first method consists in writing the numbers one beside another, and finding all their common factors by division, and then taking the product of these common factors. To illustrate, required the greatest common divisor of 42, 84, and 126.

1*st Method.*—We place the numbers one beside another as in the margin. Dividing by 2, we see that 2 is a common factor of the numbers. Dividing the quotients by 3, we see that 3 is a common factor of the numbers. Dividing these quotients by 7, we see that 7 is a common factor of the numbers; and since the final quotients 1, 2, and 3 are prime to each other, 2, 3, and 7 are all the common factors of the given numbers. Hence $2 \times 3 \times 7$ or 42, is the greatest common divisor required. This method, so far as I can learn, was published first by the author of this work, in 1855. It is now in several different text-books.

OPERATION.

$$\begin{array}{r|l} 2 & 42 \quad 84 \quad 126 \\ \hline 3 & 21 \quad 42 \quad 63 \\ \hline 7 & 7 \quad 14 \quad 21 \\ \hline & 1 \quad\; 2 \quad\; 3 \end{array}$$

$$\text{G. C. D.}=2\times 3\times 7=42$$

The second method consists in resolving the numbers into their prime factors, and taking the product of all the common factors. To illustrate, take the problem already solved by the first method.

2d Method.—Resolving the numbers into their prime factors, we find that 2, 3, and 7, are factors common to the three numbers; hence their product, which is 42, is a common divisor of the numbers; and, since these are all the common factors, 42 is the greatest common divisor.

OPERATION.

$$42=2\times 3\times 7$$
$$84=2\times 2\times 3\times 7$$
$$126=2\times 3\times 3\times 7$$
$$\text{G. C. D.}=2\times 3\times 7=42$$

CASE II. *To find the greatest common divisor when the numbers are not readily factored.* The second case may be solved by a process which may be entitled the *method of successive division.* It consists in dividing the greater number by the less, the less number by the remainder, etc., until the division terminates, the last divisor being the greatest common divisor. To illustrate, suppose it be required to find the greatest common divisor of 32 and 56.

Method.—We first divide 56 by 32, then divide the divisor, 32, by the remainder, 24; then divide the divisor, 24, by the remainder, 8, and find there is no remainder; then is 8 the greatest common divisor of 32 and 56.

OPERATION.

```
32)56(1
   32
   24)32(1
      24
       8)24(3
         24
```

This method is applicable to all numbers, and may therefore be distinguished from the methods of the previous case by naming it the *general method*, those being adapted to only a special case. A more convenient method of expressing the successive division, and one which I recommend for general adoption, is that represented in the margin. It is observed in this method that the quotients are all written in one column at the right, and that the numbers in the other columns become divisors and dividends in turn.

OPERATION.

32	56	1
24	32	1
8	24	3
	24	

Explanation.—In the explanation of the rationale of the general method of successive division, there are two distinct conceptions of the nature of the process. These two methods may, for convenience in this discussion, be entitled the *Old* and the *New* methods of explanation. By the *Old Method* of explanation I mean the one generally given in the text-books on arithmetic and algebra. The *New Method* is the one which is found in my own mathematical works. I will present each, pointing out the difference between them. Both methods are based upon the following general principles of common divisor:

1. *A divisor of a number is a divisor of any multiple of that number.*

2. *A common divisor of two numbers is a divisor of their sum, and also of their difference.*

The *Old Method* of explaining the process of successive division is briefly stated in the following propositions:

1. *Any remainder which exactly divides the previous divisor, is a common divisor of the two given quantities.*

2. *The greatest common divisor will divide each remainder, and cannot be greater than any remainder.*

3. *Therefore, any remainder which exactly divides the previous divisor is the greatest common divisor.*

Whatever the special form of the old method of explanation, and we find it considerably varied by different authors, it involves, more or less distinctly, the principles just stated

The *New Method* of conceiving of the nature of the process and explaining it, may be presented in the following principles:

1. *Each remainder is a* NUMBER OF TIMES *the greatest common divisor.*

2. *A remainder cannot exactly divide the previous divisor unless such remainder is* ONCE *the greatest common divisor.*

3. *Hence, the remainder which exactly divides the previous divisor, is* ONCE *the greatest common divisor.*

The first of these principles is evident from the consideration that a *number of times* the greatest common divisor, subtracted from *another number of times* the greatest common divisor, leaves a *number of times* the greatest common divisor.

The second of these principles becomes evident from the consideration that of any remainder and the previous divisor, the *numbers* denoting how many times the greatest common divisor is contained in each are *prime to each other;* hence, one cannot divide the other unless one of these numbers is a *unit*, or the remainder becomes *once* the greatest common divisor.

These principles may be readily seen by factoring the two numbers and then dividing. Thus, in the problem already given, knowing the greatest common divisor to be 8, we may resolve 32 and 56 into a *number of times* 8, and then divide. Observing the operations in this factored form, we see that *each remainder is a number of times the greatest common divisor*, and that the factors 7 and 4, and also 4 and 3, are respectively *prime to each other;* and also that the division terminates when we reach a divisor which is *once* the greatest common divisor, and that it

OPERATION.

$$
\begin{array}{llll}
4\times 8)7\times 8(1 & & & \\
\quad\; 4\times 8 & & & \\
\hline
& 3\times 8)4\times 8(1 & & \\
& \quad\; 3\times 8 & & \\
\hline
& & 1\times 8)3\times 8(3 & \\
& & \quad\; 3\times 8 & \\
\hline
\end{array}
$$

cannot terminate until we come to once the greatest common divisor.

In arithmetic I find it simpler to present this New Method, in a manner slightly varied from the above, preserving its spirit, but slightly changing the form to adapt it more fully to the comprehension of younger minds. To illustrate, let it be required to find the greatest common divisor of 32 and 56. Dividing as previously explained, we have the work in the margin. The explanation, showing that this process will give the greatest common divisor, is as follows:

OPERATION.

```
32)56(1
   32
   --
   24)32(1
      24
      --
       8)24(3
         24
         --
```

1st. *The last remainder*, 8, *is a number of times the greatest common divisor.* For, since 32 and 56 are each a number of times the G. C. D., their difference, 24, is a number of times the G. C. D.; and since 24 and 32 are each a number of times the G. C. D., their difference, 8, is also a number of times the G. C. D.

2d. *The last remainder*, 8, *is* ONCE *the greatest common divisor.* For, since 8 divides 24, it will divide 24+8, or 32; and since it divides 32 and 24, it will divide 24+32, or 56; and now since 8 *divides* 32 and 56, and is a *number of times* the G. C. D., and since *once* the G. C. D. is the greatest number that will divide 32 and 56, therefore 8 is *once* the G. C. D.

This second method of conceiving the subject is believed to be the true one. It is simpler than the old method, and reaches the root of the matter, which the other does not. It looks down into the process and sees the nature of the remainders, and their relation to each other. All the remainders are seen to be a number of times the greatest common divisor, each being a less and less number of times the greatest common divisor; and consequently, if the division be continued far enough, we will at length arrive at *once* the greatest common divisor. The object of dividing is thus seen to be a search for a smaller number of times the greatest common divisor, knowing that eventually we will arrive at once this factor, which will be indicated by the termination of the division. The experience of the class-room, especially in the sudden revelation of the philosophy of the division to those who thought they had a clear idea of the subject by the old method, has frequently demonstrated the superiority of the method now suggested. It is also readily seen, from this conception of the subject, that the secret of the method of finding the greatest common divisor is not in the *division* of the numbers, but in the *subtraction* of them—knowing that when we subtract one number of times a factor from another number of times the factor, the remainder is a less number of times the factor, and that the object is to continue the subtraction until we reach once the required factor.

Abbreviation.—This view of the subject leads us to discover a shorter process of obtaining the greatest common divisor than that of the ordinary method of dividing.

Thus, suppose we wish to find the greatest common divisor of 32 and 116. If we divide in the ordinary way, we will find that it requires five divisions and five quotients. If we take 4 times 32 and subtract 116 from it, we get a smaller remainder than if we subtract 3 times 32 from 116, and hence are nearer once the greatest common divisor. If we then subtract 32 from 3 times 12, we obtain a smaller remainder than if we subtract 2 times 12 from 32, and hence are nearer once the greatest common divisor, etc. This latter method requires but three multiplications and subtractions, and hence saves two-fifths of the work. In many problems nearly one-half the labor is saved by this method.

OPERATION.

32	116	4
36	128	
4	12	3
	12	3

The method of conceiving and explaining the greatest common divisor here given, is perhaps most clearly exhibited by the use of general symbols. Thus, let A and B be any two numbers, of which A is the greater; let c be their greatest common divisor, and suppose $A=ac$ and $B=bc$; then dividing the greater by the less, the smaller by the remainder, and thus continuing, we have the operation in the margin, which may be explained as follows:

$$\begin{array}{l} b.c.)\ a.\ c(q \\ \quad\ \underline{bq.\ c} \\ \quad\ (a-bq)c=r.c \\ r.c)\ b.\ c(q' \\ \quad\ \underline{rq'.\ c} \\ \quad\ (b-rq')c=r'.c \\ r'.c)\ r.\ c(q'' \\ \quad\ \underline{r'q''.c} \\ \quad\ (r-r'q'')c=r''.c \\ \quad\quad \text{etc.}\quad \text{etc.} \end{array}$$

1st. *Each remainder is a number of times the G. C. D.* This is shown by the division, since each remainder is a number of times c, the first being $(a-bq)$ times c, which we indicate by r times c, etc.

2d. *A remainder cannot exactly divide the previous divisor unless such remainder is* ONCE *the G. C. D.* To prove this it must be shown that b and r are prime to each other; also, that r and r' are prime to each other, etc. Now, if b and r are not prime to each other, they have a common factor, and hence, $r+bq$ or a contains this factor of b; but a and b are prime to each other, since c is the greatest common factor of a and b; therefore, b and r are prime to each other. In the same way it may be shown that r and r' are prime to each other, r' and r'', etc. Hence, since of two numbers prime to each other one cannot contain the other unless the latter is a unit, a remainder cannot exactly divide the previous divisor unless such remainder is once the G. C. D.

3d. Hence, *the remainder which does exactly divide the previous divisor is* ONCE *the Greatest Common Divisor.*

CHAPTER V.

THE LEAST COMMON MULTIPLE.

A MULTIPLE of a number is one or more times the number. A *Common Multiple* of two or more numbers is a number which is a multiple of each of them. The *Least Common Multiple* of several numbers is the least number which is a multiple of each of them.

This conception of a multiple is that it is a *number of times* some number. It regards the subject as a special case of forming composite numbers. A common multiple is a synthesis of all the different factors of two or more numbers, giving rise to a number which is one or more times each of those numbers. The relation of the subject to multiplication is also seen in the term *multiple* itself. The primary idea is, what number is one or more times each of several numbers?

This view of a multiple differs from that usually presented by our writers of text-books. The usual definition is—A multiple of a number is a number which exactly *contains* it. This puts *containing* as the primary idea, and makes the subject seem to originate in division rather than in multiplication. Indeed, some have gone so far in this direction as to change the name from *multiple* to *dividend*, calling it a *common dividend* instead of a common multiple. That this idea is incorrect is evident both from the term multiple, and the nature of the subject. There can be no question of the subject having its origin in multiplication, and it should certainly be defined in accordance with this view.

It will be observed that the subject of Greatest Common Divisor is placed before that of Least Common Multiple; that is, a special case of Factoring before a special case of Composition, thus reversing the general order of synthesis before analysis. The reason for this is that Common Multiple is a synthesis of factors, and in some numbers these factors are most conveniently found by the method of greatest common divisor. This order is thus a matter of convenience in performing the operation, and not that of logical relation.

Cases.—There are two general cases of Least Common Multiple, as of Greatest Common Divisor. This distinction of cases, as in the corresponding analytic process, is not founded in a variation of the general idea, but rather in the practical ease or difficulty of finding the factors of the numbers. When the numbers are readily factored we employ one method of operation; when they are not easily factored we employ another method. These two cases are formally stated as follows:

I. To find the least common multiple when the numbers are readily factored.

II. To find the least common multiple when the numbers are not readily factored.

Treatment.—The general method of treatment in the first case is to resolve the numbers into their different factors by the ordinary method of factoring, and take the product of all the different factors. In the second case, the different factors are found by the process of determining the greatest common divisor, and are then combined as before.

CASE I. *To find the least common multiple when the numbers are readily factored.* This case may be solved by two distinct methods. The first method consists in resolving the numbers into their prime factors, and then taking the product of all the different prime factors, using each factor the greatest number of times it appears in either number. Thus, required the least common multiple of 20, 30, and 70.

We first resolve the numbers into their prime factors.

Since the factors of 20 are $2 \times 2 \times 5$, the multiple must contain the factors 2, 2, and 5; since the factors of 30 are 2, 3, and 5, it must contain the factors 2, 3, and 5; and for a similar reason it must contain the factors 2, 5, and 7; hence, the least common multiple of 20, 30, and 70 must contain the factors 2, 2, 3, 5, and 7, and no others; and their product, which is 420, is the least common multiple required.

OPERATION.

$$20 = 2 \times 2 \times 5$$
$$30 = 2 \times 3 \times 5$$
$$70 = 2 \times 5 \times 7$$
$$\text{L. C. M.} = 2 \times 2 \times 3 \times 5 \times 7 = 420.$$

The second method consists in writing the numbers one beside another and finding all the different factors by division, and then taking the product of these factors. To illustrate, find the least common multiple of 24, 30, and 70.

Placing the numbers beside one another, and dividing by 2, we find that 2 is a factor of all the numbers; it is therefore a factor of the least common multiple. Dividing the quotients by 3, we see that 3 is a factor of some of the numbers; it is therefore a factor of the least common multiple. Continuing to divide, we find all the different factors of the numbers to be 2, 3, 4, 5, and 7; hence, their product, which is 840, will be the least common multiple required.

OPERATION.

```
2)24 30 70
3)12 15 35
5) 4  5 35
   4  1  7
```

CASE II. *To find the least common multiple when the numbers are not readily factored.* The second case is solved by a method which may be called the method of greatest common divisor. By it, when there are two numbers, we find the greatest common divisor of the two numbers and multiply one of them by the quotient of the other divided by their greatest common divisor. When there are more than two numbers, we find the least common multiple of two of the numbers, and then of this multiple and the third number, etc. To illustrate, required the least common multiple of 187 and 221.

We first find the greatest common divisor to be 17. Now, the least common multiple of 187 and 221 must be composed of all the factors of 187, and all the factors of 221 not contained in 187. If we divide 221 by the greatest common divisor, we shall obtain the factors of 221 not belonging to 187; hence, the least common multiple is equal to $187 \times 221 \div 17$, which we find is 2431.

OPERATION.

187	221	1
170	187	
17	34	5
	34	2

$$\text{L. C. M.} = 187 \times \frac{221}{17} = 2431$$

Another statement for this method is, *divide the product of the two numbers by their greatest common divisor.* The value of this method may be seen by attempting to find the least common multiple of 1127053 and 2264159 by each method.

This method is very clearly exhibited by the following general explanation. Let A and B be any two quantities, and let their greatest common divisor be represented by c, and the other factors by a and b, respectively; then we shall have the L. C. M.$=a \times b \times c$, Case I.; but $b \times c = B$, and $a = \frac{A}{c}$; hence, L. C. M. $= \frac{A}{c} \times B$.

OPERATION.

$$A = a \times c$$
$$B = b \times c$$
$$\text{L. C. M.} = a \times b \times c = \frac{A}{c} \times B$$

CHAPTER VI.

INVOLUTION.

INVOLUTION is the process of forming composite numbers by the synthesis of equal factors. It is, as has been previously explained, a special case of *Composition*. If in the general synthesis of factors, we fix upon the condition that all the factors are to be equal, the process is called *Involution*, and the composite number formed is called a *Power* of that factor.

Involution may, therefore, be defined as the process of raising numbers to required powers. The *power* of a number is the product obtained by using the number as a factor any number of times. The different powers of a number are called, respectively, the *square*, the *cube*, the *fourth power*, etc. The square of a number is the product obtained by using the number as a factor twice. The cube of a number is the product obtained by using the number as a factor three times. These definitions, which are beginning to be adopted by authors, are regarded as improvements upon those framed from the usual conception of the subject.

Symbol.—The power of a quantity is indicated by a figure written at the right, and a little above the quantity. Thus, the third power of 5 is indicated by 5^3. The earlier writers on mathematics denoted the powers of numbers by an abbreviation of the name of the power. *Harriot*, an eminent mathematician of the 16th century, repeated the quantity to indicate the power; thus, for *a* fourth power he wrote *aaaa*. The present convenient system of exponents was introduced by

Descartes, an eminent philosopher and mathematician celebrated for his "*cogito, ergo sum*," and the invention of the method of *Analytical Geometry*.

Cases.—To raise a number to each different power is a variation of the general idea, and might be regarded as presenting distinct cases; but the methods of operation in each one of these cases are so similar, that they may all be considered under one head. In raising a number to a given power, we may have two objects in view:—first, merely to find the required power; and second, to ascertain the law by which the different parts of a number, as expressed in the Arabic system, are involved. These two objects require different methods of procedure, and upon this difference of method we may found two distinct cases of involution. In practice, it is convenient to divide the second case into the consideration of the square and the cube, thus making three cases. These cases, formally expressed, are as follows:

I. To raise a number to any required power.

II. To raise a number to the second power, and ascertain the law by which the power is formed.

III. To raise a number to the third power, and ascertain the law by which the power is formed.

Treatment.—The general method of treatment is to involve the factors by multiplication. In the first case a variation occurs for the purpose of abbreviation, giving two methods. In the second and third cases the number is resolved into parts and involved in two different ways, giving also two distinct methods. The treatment of both of these cases will now be presented.

CASE I. *To raise a number to any required power.* This case may be solved by forming a product by using the number as a factor as many times as there are units in the index of the power. Thus, to find the third power of 4, we multiply 4 by 4 giving 16, and then multiply 16 by 4

OPERATION.

$$\begin{array}{r} 4 \\ 4 \\ \hline 16 \\ 4 \\ \hline 64 \end{array}$$

giving 64, which is the cube of 4, since the number is used as a factor three times.

In all powers higher than the cube, we may abbreviate the process by taking the product of one power by another. Thus, in finding the 8th power of 2, we may first find the square of 2, which is 4, then multiply 4, the square, by itself, obtaining 16, the 4th power of 2, and then multiply 16, the 4th power, by itself, giving 256, the 8th power of 2. This method may be applied to all powers higher than the third, and is much more convenient in practice. Thus, in finding the 5th power, we may take the product of the 2d and 3d powers, or the product of the square by the square by the first power; in finding the 6th power, we may cube the 2d power, or square the 3d power, or multiply the 4th power by the square, etc.

OPERATION.

$$\begin{array}{r} 2 \\ 2 \\ \hline 4 \\ 4 \\ \hline 16 \\ 16 \\ \hline 256 \end{array}$$

CASE II. *Squaring Numbers and finding the law.* This case may be solved by two distinct methods. The first consists in separating the number into its elements of units, tens, etc., and multiplying as in *algebra* so as to exhibit the law by which the parts are involved. The second method performs the process of involution as determined by the building up of a figure in *geometry*. These two methods may be distinguished as the *algebraic* and *geometric*, or the *analytic* and *synthetic* methods. The ultimate object of these methods is to derive a law of involution by which we may be able to derive methods of evolution. These two methods apply both to the squaring and cubing of numbers. The synthetic method cannot be extended beyond the cubing of numbers; the analytic method is general and will apply to all powers, but is of no practical use in arithmetic beyond the cube. We will, therefore, apply these two methods only to the squaring and cubing of numbers.

ANALYTIC METHOD.—By the *Analytic Method* of squaring numbers, we separate the number into its units, tens, etc., and keep these elements distinct in the involution of the number, so that the law of the process becomes apparent. To illustrate, find the square of 25.

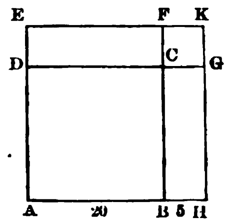
Twenty-five equals $20+5$, or 2 tens and 5 units. Writing the number as in the margin, and multiplying by 5 and by 20, and taking the sum of these products, we have $20^2+2\ (5\times20)+5^2$. From this we see that the square of a number consisting of *tens* and *units*, equals the *tens*^{2}+2 *times tens*$\times$*units*+*units*^{2}.

OPERATION.

$$\begin{array}{r} 20+5 \\ 20+5 \\ \hline 5\times20+5^2 \\ 20^2+5\times20 \quad \\ \hline 20^2+2(5\times20)+5^2 \end{array}$$

If we involve in the same manner a number consisting of hundreds, tens, and units, we shall find the following law: The square of a number consisting of hundreds, tens, and units equals *hundreds*^{2}+2$\times$*hundreds*$\times$*tens*+*tens*^{2}+2$\times$(*hundreds*+*tens*)$\times$*units*+*units*^{2}.

SYNTHETIC METHOD.—The *Synthetic Method* of solving the same problem is as follows: Let the line AB represent a length of 20 units, and BH, 5 units. Upon AB construct a square: the area will be $20^2=400$ square units. On the two sides DC and BC construct rectangles each 20 units long and 5 broad, the area of which will be $5\times20=100$, and the area of both will be $2\times100=200$ square units. Now add the little square on CG, whose area is $5^2=25$ square units, and the sum of the different areas, $400+200+25=625$, is the area of a square whose side is 25.



When there are three figures, after completing the second square as above, we must make additions to it as we did to the first square. When there are four figures there are three additions, etc.

CASE III. *Cubing Numbers to find the law.* This case may also be solved by two distinct methods, as in squaring numbers, which we distinguish as the *analytic* and *synthetic* methods. The former involves the number by the method of algebra; the latter by the principles of geometry.

ANALYTIC METHOD.—By the *Analytic Method* we resolve the number into its elements of units, tens, etc., and keep it in this form as we perform the process of involution, that we may exhibit the law by which the elements of a number enter into its cube. To illustrate, find the third power of 25.

Resolving the number into its units and tens and squaring as above, we have $20^2+2(5\times20)+5^2$. Multiplying the square by 5 and then by 20, and taking the sum of the products, we have the cube of 25, as given in the margin. Examining the result, we see that the cube of a number of two digits equals $tens^3+3\times tens^2\times units+3\times tens\times units^2+units^3$.

OPERATION.

$$\begin{array}{r} 25^2=20^2+2(5\times20)+5^2 \\ 20+5 \\ \hline 5\times20^2+2\times5^2\times20+5^3 \\ 20^3+2\times5\times20^2+5^2\times20 \\ \hline 20^3+3\times5\times20^2+3\times5^2\times20+5^3 \end{array}$$

Cubing a number of three digits, we obtain the following law: The cube of a number of three digits equals $hundreds^3+3\times hundreds^2\times tens+3\times hundreds\times tens^2+tens^3+3\times(hundreds+tens)^2\times units+3\times(hundreds+tens)\times units^2+units^3$.

SYNTHETIC METHOD.—By the *Synthetic Method* we use a cube to determine the process of involution. To illustrate, let us find the cube of 45 by this method.

Let A, Fig. 1, represent a cube whose sides are 40 units; its contents will be $40^3=64000$. We then wish to increase the size of this cube so that its sides will be 45 units. To increase its dimensions by 5 units, we must add first the three rectangular slabs, B, C, D, Fig. 2; 2d, the three corner pieces, E, F, G, Fig. 3; 3d, the little cube H, Fig. 4. The three slabs B, C, D, are 40 units long and wide and 5 units thick; hence, their contents are $40^2\times5\times3=24000$; the contents of the corner pieces, E, F, G, Fig. 3, whose length is 40 and breadth and thickness 5, equal $40 \times 5^2 \times 3 = 3000$, and the contents of the little cube H, Fig. 4, equal $5^3 = 125$; hence the contents of the cube represented by Fig. 4 are $64000 + 24000 + 3000 + 125 = 91125$. Therefore, the cube of 45, etc.

Fig. 1.

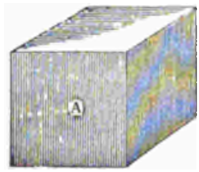
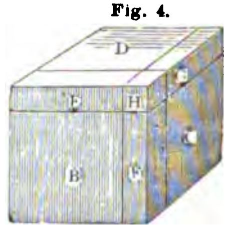


Fig. 2.

Fig. 3.

Fig. 4.



Here we see that 40^3 is the *cube of the tens;* $40^2 \times 5 \times 3$ is $tens^2 \times units \times 3$; $40 \times 5^2 \times 3$ is $3 \times tens \times units^2$; and 5^3 is $units^3$; hence we have, as before, the cube of a number of *tens* and *units* equals $tens^3 + 3 \times tens^2 \times units + 3 \times tens \times units^2 + units^3$.

OPERATION.

$$40^3 = 64000$$
$$40^2 \times 5 \times 3 = 24000$$
$$40 \times 5^2 \times 3 = 3000$$
$$5^3 = 125$$
Hence, $45^3 = 91125$

When there are three figures in the number, we complete the second cube as above, and then make additions and complete the third in the same manner. If there are still some figures, and no more blocks to make additions, let the first cube represent the cube already found, and then proceed as at first.

CHAPTER VII.

EVOLUTION.

EVOLUTION is the process of finding one of the several equal factors of a number. It is an analytic process, the converse of the process of Involution. Involution is a synthesis of equal factors; Evolution is an analysis into equal factors. The former is a special case of composition; the latter is a special case of factoring. One finds its origin in multiplication; the other in division. Both are contained in the primary synthetic and analytic ideas, and are the result of pushing forward and specializing those notions.

Any one of the several equal factors of a number is called a *root* of that number. The *degree* of a root depends upon the number of equal factors. The *square root* of a number is *one* of its *two* equal factors. The *cube root* of a number is *one* of its *three* equal factors, etc. These definitions are regarded as an improvement upon the old ones, that the square root of a number is a number which multiplied by itself will produce the number, and similarly for the other roots. Evolution may also be defined as the process of finding any required root of a number.

Symbol.—The *Symbol* of Evolution is $\sqrt{}$, called the radical sign. This sign was introduced by *Stifelius*, a German mathematician of the 15th century. It is a modification of the letter *r*, the initial of *radix*, or root. Formerly, the letter *r* was written before the quantity whose root was to be extracted, and this gradually assumed its present form, $\sqrt{}$.

To indicate the degree of the root to be extracted, a figure is prefixed to the radical sign; thus, $\sqrt[2]{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, etc., denote respectively the square root, cube root, fourth root, etc. This figure is called the *index* of the root, because it indicates the root required. The index of the square root is usually omitted, perhaps because the symbol was applied to the square root for some time before its use was extended to the higher roots. The roots of numbers are also indicated by fractional exponents; as $4^{\frac{1}{2}}$, $8^{\frac{1}{3}}$, etc.

Cases.—Each different root might be regarded as constituting a distinct case, but it is most convenient to treat the subject under three general cases, as in Involution. These three cases correspond to those of Involution, and may be formally expressed as follows:

I. To extract any root of a number when it can be conveniently resolved into its prime factors.

II. To extract the square root of a number when it can not be conveniently factored.

III. To extract the cube root of a number when it can not be conveniently factored.

Treatment.—The general method of treatment is to analyze the number into the parts required. In the first case, we analyze the number into its prime factors, and then make a synthesis of some of these factors. In the second and third cases, we separate the number into parts by several distinct methods, corresponding to those of Involution.

CASE I. *To extract any root when the number can be readily factored.* This case is solved by resolving the number into its prime factors, and then involving the factors so as to obtain the equal factor required. For the square root we take the product of *each* of the *two* equal factors; for the cube root we take the product of *each* of the *three* equal factors, etc.

Thus, to find the square root of 1225, we resolve the number into its prime factors, 5, 5, 7, 7, and take the product of *one* of the two 5's and *one* of the two 7's, giving us 5×7, or 35.

OPERATION.

$1225 = 5 \times 5 \times 7 \times 7$

Sq. rt. $= 5 \times 7 = 35$

To find the cube root of 1728 we resolve the number into its prime factors, as shown in the margin, and take the product of *one* of the *three* 3's, and *one* of the *three* 4's, giving 3×4, or 12. In a similar manner we find any root of any perfect power that can be resolved into its prime factors.

OPERATION.

$1728=3\times3\times3\times4\times4\times4$
Cu. rt.$=3\times4=12$

CASE II. *To extract the Square Root of a number.* The *Square Root* of a number is one of the two equal factors of the number. The square root of a number may also be defined to be a number which, used as a factor twice, will produce the given number. The former definition is somewhat analytic; the process of thought is from the number to its elements. The latter is rather synthetic; the process of thought is from the elements to the number.

The method of extracting the square root of a number consists in analyzing the number into two equal multiplicative parts. This is done by first finding the highest term of the root, taking its square out of the number, and using it, according to the laws of involution, to determine the next term of the root, etc. The method being found in all the works on arithmetic, need not be stated here.

Explanation.—There are two methods of deriving the rule for square root, or of explaining the reason for the operation. These methods are distinguished as the *Analytic* and *Synthetic* methods. The former consists in resolving the number into its elements by the laws obtained by the analytic method of involution; the latter consists in finding the root by means of a geometrical diagram by reversing the process of the corresponding method of involution. The synthetic method will apply to both the square and cube root of numbers, but cannot be extended beyond the cube root. The analytic method is general, and can be applied to the determining of any root of a number.

In order to determine how many figures there are in the root, and where to begin the extraction of the root, we employ the following principles:

1. *The square of a number consists of twice as many figures as the number, or of twice as many less one.*

This principle may be demonstrated as follows: Any integral number between 1 and 10 consists of one figure, and any number between their squares, 1 and 100, consists of *one* or *two* figures; hence the square of a number of *one* figure is a number of *one* or *two* figures. Any number between 10 and 100 consists of *two* figures, and any number between their respective squares, 100 and 10,000, consists of *three* or *four* figures; hence, the square of a number of *two* figures is a number of *three* or *four* figures, etc. Therefore, etc.

$$1^2 = 1$$
$$10^2 = 100$$
$$100^2 = 10,000$$
$$1000^2 = 1,000,000$$

2. *If a number be pointed off into periods of two figures each, beginning at units place, the number of full periods, together with the partial period at the left, if there be one, will equal the number of places in the square root.*

This is evident from Prin. 1, since the square of a number contains *twice* as many places as the number, or *twice* as many less one.

ANALYTIC METHOD.—By the analytic method of explaining the process of extracting the square root of numbers, we resolve the number into its elements, and derive the method of operation by knowing the law of the synthesis of these elements. It is appropriately named the analytic method, because it analyzes a number into its elements, and operates by reversing the synthetic process of involution. We will illustrate this method by extracting the square root of 625.

Explanation.—By the principles of involution we see that there will be two figures in the root, hence the number consists of the square of the *tens* plus the *units* of the root, which equals the *square of the tens, plus twice the tens into the units, plus the square of the units.* The greatest number of

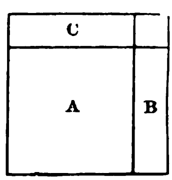
$$t^2+2tu+u^2=6\cdot25(25$$
$$t^2=20^2\quad=400$$
$$2tu+u^2\quad=225$$
$$2t=40$$
$$u=5$$
$$2tu+u^2=225$$

tens whose square is contained in 625 is 2 tens; squaring the tens and subtracting we have 225, which equals *twice the tens into the units, plus the square of the units.* Now, since *twice the tens into the units* is usually much greater than the *units squared,* 225 consists principally of *twice the tens into the units;* hence if we divide 225 by *twice* the tens, we can ascertain the units. Twice the tens equals 20×2, or 40; dividing 225 by 40, we find the units to be 5, etc.

In the margin the law of the involution of the elements is shown by the use of the letters t and u, the initials of *tens* and *units.* This representation of the law of the formation of the number enables us to separate it into its elements.

SYNTHETIC METHOD.—By the synthetic method we use a geometrical figure and derive the process from the method of forming a square whose area shall equal the given number. It is called synthetic because we commence with a smaller square and add parts to it, until we find a square of the required area. The method of forming the square will give us a method of finding the square root. To illustrate, let it be required to extract the square root of 625.

Explanation.—The greatest number of tens whose square is contained in 625 is 2 tens. Let A, Fig. 1, represent a square whose sides are 2 tens or 20 units, its area will be the square of 20, or 400. Subtracting 400 from 625, we have 225, hence our square is not large enough by 225; we must therefore increase it by 225. To do this we add the two rectangles B and C, each of which is 20 units long, and since they nearly complete the square, their area must be nearly 225 units; hence, if we divide by their length we can find their width. Their length is 20×2=40, hence their width is 225÷40 or 5



Now complete the square by the addition of the little corner square whose side is 5 units, and then the entire length of all the additions is 40+5, or 45 units, and multiplying by the width we find their area to be 225 square units. Subtracting, nothing remains; hence, the side of a square which contains 625 square units is 25 units.

The same method will apply when there are more than two figures in the root. The methods of operation indicated by both the analytic and synthetic methods of explanation, are the same. These methods give the usual rule for the extraction of the square root.

CASE III. *The Cube Root of Numbers.* The *Cube Root* of a number is one of the three equal factors of the number. The cube root of a number may also be defined to be a number which, used as a factor three times, will produce the given number. Again, the cube root of a number may be defined as a number which, raised to the third power, will produce the given number. These definitions are all correct, though they differ in idea. The first is analytic; the thought is from the number to its elements. The second and third are synthetic; the process of thought is from the elements to the number.

The method of extracting the cube root of a number consists in analyzing it and finding one of its three equal multiplicative parts. This is done by first finding the highest term of the root and taking its cube out of the number, then finding the second term by means of the first term, taking their combination out of the number, etc. There are several methods of doing this, the three most important of which may be distinguished as the *Old Method*, a *New Method*, and *Horner's Method*. There are several other methods, which I do not regard of sufficient importance to consider in this work.

Old Method.—The Old Method is so called because it is the one which has for a long time been taught and practiced. It may be distinguished by the use of 300 and 30 in finding trial and complete divisors. By a slight modification of the method the ciphers of these multipliers may be omitted, and this form of the method is now generally preferred. The method may be stated as follows:

RULE.—I. *Separate the number into periods of three figures each, beginning at units place.*

II. *Find the greatest number whose cube is contained in the left-hand period; place it at the right and subtract its cube from the period, and annex the next period to the remainder for a dividend.*

III. *Take 3 times the square of the first term of the root regarded as tens for a* TRIAL DIVISOR; *divide the dividend by it, and place the quotient as the second term of the root.*

IV. *Take 3 times the last term of the root multiplied by the preceding part regarded as tens; write the result under the trial divisor, and under this write the square of the last term of the root; their sum will be the* COMPLETE DIVISOR.

V. *Multiply the* COMPLETE DIVISOR *by the last term of the root; subtract the product from the dividend, and to the remainder annex the next period for a new dividend. Take 3 times the square of the root now found, regarded as tens, for a trial divisor, and find the third term of the root as before; and thus continue until all the periods have been used.*

Explanation.—This process of extracting cube root may be explained by two distinct methods, distinguished as the *analytic* and *synthetic* methods. The analytic method consists in resolving the number into its elements by the laws obtained from the analytic method of involution. The synthetic method consists in ascertaining the different terms of the root by the building up of a geometrical cube.

In order to determine the number of figures in the root and with what part of the number to begin the evolution, it is necessary to state and demonstrate the following principle:

1. *The cube of a number consists of three times as many figures as the number, or of three times as many less one or two.*

This principle may be demonstrated as follows: Any integral number between 1 and 10 consists of one figure, and any integral number between their cubes, 1 and 1000, consists of one, two, or three figures; hence the *cube* of a number of *one* figure is a number of *one*, *two*, or *three* figures.

$$1^3=1$$
$$10^3=1000$$
$$100^3=1,000,000$$

Any number between 10 and 100 consists of *two* figures, and any number between their cubes, 1000 and 1,000,000, consists of *four*, *five*, or *six* figures; hence the cube of a number of *two* figures consists of *three times two* figures, or *three times two*, less *one* or *two* figures.

2. *If a number be pointed off into periods of three figures each, beginning at units place, the number of full periods together with the partial period at the left, if there be one, will equal the number of figures in the root.*

This is evident from Prin. 1, since the cube of a number contains *three times* as many places as the number, or *three times* as many, *less one or two.*

ANALYTIC METHOD.—By the analytic method of explaining the process of extracting the cube root of numbers, we resolve the number into its elements and derive the process by knowing the law of the synthesis of these elements in the process of involution. We will illustrate the method by the solution of the following problem: Required the cube root of 91125.

Solution.—Since the cube of a number consists of *three times as many places as the number* itself, or of three times as many less *one* or *two*, the cube root of 91125 will consist of two places, and hence consist of tens and units, and the given number will consist of the *cube of the tens, plus three times the square of the tens into the units, plus three times the tens into the square of the units, plus the cube of the units.*

	91·125(40	
$40^3=$	64 000	5
$40^2\times3=4800$	27125	45
$40\times5\times3=\ \ 600$		
$5^2=\ \ \ 25$		
5425	27125	

The greatest number of tens whose cube is contained in the given number is 4 tens. Cubing the tens and subtracting, we have 27125, which equals *three times the square of the tens into the units*, etc. Now, since three times the square of the tens into the units is much greater than all the rest of the expression, 27125 must consist principally of *three times the square of the tens into the units;* hence if we divide by three times the square of the tens we can ascertain the units. Three times the tens squared equals $3\times40^2=4800$; dividing by 4800 we find the units to be 5. We then find *three times the tens into the units* equal to $40\times5\times3=600$, and *units squared* equals $5^2=25$. Taking the sum and multiplying by the *units*, we have 27125, and subtracting, nothing remains. Hence the cube root of 91125 is 45. From this solution we readily derive the rule given above.

Synthetic Method.—By the synthetic method of explanation we use a geometrical figure, a cube, and derive the process from the method of forming a cube whose contents shall equal the number of units in the given number. The number is regarded as expressing the number of cubic units in a cubical block, the number of linear units in whose side will be the cube root of the number. It is appropriately called synthetic, since we begin with a cube and add parts to it until we find a cube of the required contents. The method of forming the cube indicates the process of finding the cube root. This method may be illustrated by the solution of the problem already given: Required the cube root of 91125.

Solution.—We find the number of figures in the root as before, and then proceed as follows: The greatest number of tens whose cube is contained in the given number is 4 tens.

Let A, Fig. 1, represent a cube whose sides are 40, its contents will be $40^3=64000$. Subtracting from 91125 we find a remainder of 27125 cubic units; hence, the cube A is not large enough to contain 91125 cubic units by 27125 cubic units; we will therefore increase it by 27125 cubic units.

Fig. 1.

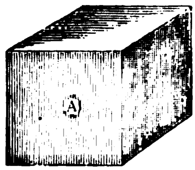


Fig. 2.

Fig. 3.

Fig. 4.

To do this we add the three rectangular slabs B, C, D, Fig. 2, each of which is 40 units in length and breadth; and since they nearly complete the cube, their contents must be nearly 27125; hence, if we divide 27125 by the sum of the areas of one of their faces as a base, we can ascertain their thickness.

OPERATION.

	91·125(40	
$40^3=$64 000	5	
$40^2\times3=$4800	27125	45
$40\times5\times3=$ 600		
$5^2=$ 25		
5425	27125	

The area of a face of one slab is $40^2=1600$, and of the three, $3\times1600=4800$; and dividing 27125 by 4800 we have a quotient of 5; hence the thickness of the additions is 5 units. We now add the three corner pieces E, F, and G, each of which is 40 units long, 5 wide, and 5 thick; hence the surface of a face of each is $40\times5=200$ square units, and of the three it is $200\times3=600$ square units.

We now add the little corner cube H, Fig. 4, whose sides are 5 units, and the surface of a face is $5^2=25$. We now take the sum of the surfaces of the additions, and multiply this by the common thickness, which is 5, and we have their solid contents equal to $(4800+600+25)\times5=27125$. Subtracting, nothing remains; hence the cube which contains 91125 cubic units is 40+5 or 45 units on a side.

When there are more than two figures we increase the size of the new cube, Fig. 4, as we did the first, or let the first cube, Fig. 1, represent the new cube, and proceed as before.

THE METHODS COMPARED.—These two methods of explaining the process of extracting the square and cube roots of numbers are entirely distinct; they are based upon different ideas, though they give rise to the same practical operation. The synthetic method is the one generally given in the text-books on arithmetic; the analytic method was, until recently, confined to algebra. It has been a question which of these methods of explanation is the better, some preferring the one and some the other. In my own opinion the analytic method is to be preferred for several reasons, among which the following may be stated:

First, it is in accordance with the genius of arithmetic; we explain an arithmetical subject upon arithmetical principles. By the synthetic method we leave the subject of arithmetic, and bring in geometry to explain arithmetic. Should it be said in reply that by the analytic method we are explaining arithmetic by algebra, let it be remembered that algebra has been called "universal arithmetic," and that all the algebra that is here used is purely arithmetical. In other words, though we may indicate the analysis of the number by letters, the idea is purely an arithmetical one, and is in no way dependent upon the principles of algebra as different from arithmetic.

Second, I hold that a full, complete, and thorough insight into the subject can be obtained only by the analytic method. The geometric method indicates the process, as well as the analytic; but the analytic method shows the nature of the process, it exhibits the law of the formation of the square or cube as a pure process of arithmetic; and this gives a deeper insight into the subject than can be obtained by the other method. One who knows evolution only by the synthetic method, does not know it thoroughly.

Third, the analytic method is *general;* it will explain the method of extracting all roots. The geometrical method is *special;* it enables us to extract the square and cube roots only. Thus, the square root is regarded as the side of a square, the cube root as the side of a cube; but we have no geometrical conception of the fourth root, no figure corresponding to the fourth power, and therefore no idea of a fourth root; and so on for the higher roots.

In respect of the comparative difficulty of the two methods, it may be remarked that it is generally supposed that the synthetic method is much easier than the analytic. This, however, I very much doubt; and this opinion is founded, not only upon theory, but also upon the experience of those who have tried both methods. I believe that a thorough knowledge of the subject can be gained much sooner by the analytic than by the synthetic method. My observation has been that pupils often are able to run over the geometrical explanation without really understanding it. It is, therefore, recommended that the analytic method be introduced into our text-books and systems of instruction.

Fig. 1.

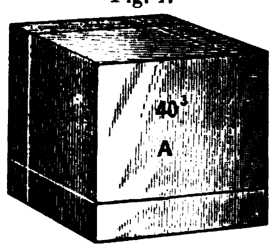
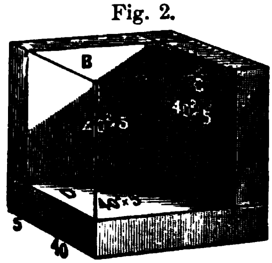


Fig. 2.



The so-called synthetic methods of evolution may also be presented in an *analytic* form. Thus, instead of adding to the square A, page 271, we can begin with the large square, take out the square A, then obtain the width of the rectangles and the dimensions of the corner square, and then subtract. Indeed, this seems the more natural method, and is now being adopted by American writers. When thus presented, it would be better to call the two methods the *algebraic* and *geometric* methods.

The same may be illustrated in the extraction of the cube root. Let Fig. 1 represent a cube which contains 91125 cubic units. Taking out the cube, A ($40^3 = 64000$), we have a solid, Fig. 2, representing 27125 cubic units. This solid consists principally of the three slabs, B, C, and D, each 40 units in length and breadth. Dividing 27125 by the sum of the areas of a face of each, ($3 \times 40^2 = 4800$), we find their thickness is 5 units. Removing the slabs, there remain three solids, Fig. 3, each 40 units by 5 units, hence the surface of a face of the three is $3 \times 40 \times 5 = 600$ square units.

Fig. 3.

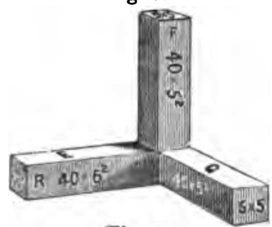
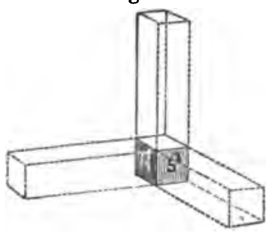


Fig. 4.



Removing E, F, and G, there remains the small cube H, Fig. 4, the surface of one of whose faces is $5^2 = 25$ square units. Multiplying the sum of all these surfaces by the common thickness, 5, we have $(4800 + 600 + 25) \times 5 = 27125$ cubic units.

New Method of Cube Root.—I will now present a method of extracting cube root which is much simpler and more convenient than the ordinary one, and indeed than any other with which I am acquainted. This method seems to have been approximated by several writers, although I have not found any who present it in the form in which it is here given. Its arrangement in columns will remind one of Horner's Method, but a very slight inspection will show that it is quite different from it.

In order to present the method in a manner to be most readily understood, I will first solve a problem, and then state the rule. In the operation I will indicate the *trial divisor* by t. d., and the complete divisor by C. D., and use dots, thus, . . to indicate the local value of the terms. The reason for the method of obtaining the trial and complete divisors may be readily shown by the formula.

1. Extract the cube root of 14706125.

Solution.—We find the number of figures in the root, and the first term of the root, as in the preceding method.

OPERATION.

1st Col.	2d Col.	
2	12. . t. d.	14·706·125(245
4	256	8
64	1456 c. d.	6706
8	16	5824
725	1728. . t. d.	882125
	3625	
	176425 c. d.	882125

We write 2, the first term of the root, at the left at the head of Col. 1st; 3 times its square with two dots annexed, at the head of Col. 2d; its cube under the left-hand period; then subtract and annex the next period for a dividend; and divide it by the number in Col. 2d, as a *trial divisor*, for the second term of the root.

We then take 2 times 2, the first term, and write the product, 4, in Col. 1st, under the 2, and add; then annex the second term of the root to the 6 in Col. 1st, making 64, and multiply 64 by 4 for a *correction*, which we write under the trial divisor; and adding the *correction* to the *trial divisor*, we have the *complete divisor*, 1456. We then multiply the complete divisor by 4, subtract the product from the dividend, and annex the next period for a new dividend.

We then square 4, the second figure of the root, write the *square* under the *complete divisor*, and add the *correction*, the *complete divisor* and the *square* for the next *trial divisor*, which we find to be 1728. Dividing by the trial divisor we find the next term of the root to be 5.

We then take 2 times 4, the second term, write the product 8 under the 64, add it to 64, and annex the third term of the root to the sum, 72, making 725, and then multiply 725 by 5, giving us 3625 for the next *correction*. We then find the *complete divisor* by adding the *correction* to the *trial divisor;* multiply the true divisor by 5, and subtract and have no remainder.

The reason for this method will appear by using letters to show the law of finding the trial and complete divisors. The object is to find a general method of obtaining these divisors, so that any one divisor may be used in forming the next following divisor. It is evident that by such a method the work will be greatly abridged.

The rule drawn from this explanation may be stated as follows:

RULE.—I. *Separate the number into periods of three figures each; find the greatest number whose cube is contained in the first period, and write it in the root.*

II. *Write the first term of the root at the head of 1st Col., 3 times its square with two dots annexed at the head of 2d Col., and its cube under the first period; subtract and annex the next period to the remainder for a dividend, divide by the number in 2d Col. as a* TRIAL DIVISOR, *and place the quotient as the second term of the root.*

III. *Add twice the first term of the root to the number in the first column; annex the second term of the root, multiply the result by the second term, and write the product under the trial divisor for a* CORRECTION; *add the* CORRECTION *to the* TRIAL DIVISOR, *and the result will be the* COMPLETE DIVISOR; *multiply the* COMPLETE DIVISOR *by the last term of the root, subtract the product from the dividend, and annex the next period to the result for a new dividend.*

IV. *Square the last term of the root, and take the sum of this* SQUARE, *the last* COMPLETE DIVISOR *and the last* CORRECTION, *and annex two dots, for a new* TRIAL DIVISOR; *divide the dividend by it and obtain the next term of the root.*

V. *Add twice the seeond term of the root to the last number in the first column; annex the last term of the root to the sum, multiply the result by the last term, and write the product under the last trial divisor for a* CORRECTION; *add the* CORRECTION *to the* TRIAL DIVISOR, *and the result will be the* COMPLETE DIVISOR; *use this as before, and thus continue until all the periods have been used.*

A part of this method can be easily remembered by means of the following formulas, which show the formation of the *trial* and *complete divisors:*

1. Trial Divisor+Correction=Complete Divisor.
2. Correction + Complete Divisor + Square=Trial Divisor.

To show the application of the method we will extract the cube root of 41673648563.

1st Col.	2d Col.	
3	27 . . t. d.	41·673·648·563(3467
6	376	27
94	3076 c. d.	14673
8	16	12304
1026	3468 . . t. d.	2369648
12	6156	
10387	352956 c. d.	
	36	2117736
	359148 . . t. d.	251912563
	72709	
	35987509 c. d.	251912563

Horner's Method.—*Horner's Method* of extracting the cube root was derived from a method of solving cubic equations invented by Mr. Horner, of Bath, England. It was first published in the *Philosophical Transactions* for 1819; under the title of "A New Method of Solving Numerical Equations of all orders by Continuous Approximations." Its inventor, Mr. W. G. Horner, was a teacher of mathematics at Bath; he died in 1837. It is considered one of the most remarkable additions made to arithmetic in modern times. DeMorgan says that the first elementary writer who saw the value of Horner's method was J. R. Young, who introduced it in an elementary treatise on algebra, published in 1826. Among the first to introduce it into arithmetic in this country was Prof. Perkins, of New York.

This method differs from both of those already explained, and possesses merits which strongly recommend it for general adoption. It is very concise—the root of a large number can be extracted with one-half of the work required by the old method. Its conciseness arises from the fact that it proceeds upon a principle which enables us to make use of work already obtained, while the old method requires new calculations every time we find a trial or true divisor. In other words, it is an organized method by which the work is so economized that no operations are superfluous, but each result obtained is made use of in obtaining a subsequent result.

It is entirely general in its character, applying to the extraction of all the higher roots. This method can be explained both analytically and synthetically. It is presented in several of the higher arithmetics, and need not be stated here. It is more difficult to remember than either of the other methods, and this is perhaps the principal objection to its general adoption. The "New Method" for cube root—it does not apply to higher roots—is, however, preferred to Horner's, being quite as concise, and much more readily acquired and remembered.

APPROXIMATE ROOTS.—The invention of rules for approximating to the square and other roots of numbers, where those roots are surds, was a favorite speculation with earlier writers on arithmetic and algebra. These rules will be most readily understood and their relative values seen by stating them in algebraic language.

1. The rule given by the Arabs is expressed by the formula,

$$\sqrt{(a^2+x)}=a+\frac{x}{2a}$$

This approximation gives the root in excess; but to increase its accuracy, we may repeat the process, making use of the root obtained. This is the rule given by Lucas di Borgo, and subsequently by Tartaglia, who derived it in common with the rest of his countrymen from Leonard of Pisa.

2. The rule given by Juan de Ortega, 1534, is expressed by the following formula:

$$\sqrt{(a^2+x)}=a+\frac{x}{2a+1}$$

This approximation is in defect, but, generally speaking, more accurate than the former.

3. The third method of approximation was proposed by Orontius Fineus, Professor of mathematics in the university of Paris, and who long enjoyed an uncommon reputation in consequence of his having introduced the knowledge of the mathematics of Italy among his countrymen. His method consisted in adding 2, 4, 6, or any even number of ciphers to the number whose root was required, and then reducing the number expressed by the additional figures of the root resulting from these ciphers, to sexagesimal parts of an integer. Thus, in extracting the square root of 10, he would get 3 | 162, which reduced to sexagesimals, became 3. 9′. 43″. 12‴.

This is the most remarkable approximation to the invention of decimals which preceded the age of Stevinus. If the author had stopped short at the first separation of the digits in the root, it would have expressed the square root of 10 to 3 decimal places; but the influence of the use of sexagesimals diverted him from this very natural extension of the decimal notation, and retarded for more than half a century this improvement in the science of numbers

The method of Fineus excited the attention of contemporaneous mathematicians, who in adopting it, however, did not reduce the result to sexagesimals, but merely subscribed, as a denominator to the whole not considered as integral, 1 with half as many ciphers as had been added in the operation, giving $\sqrt{10}=\frac{3162}{1000}$. It is under this form that it is noticed by Tartaglia and Recorde. Pelletier also, a pupil of Orontius Fineus, after noticing the second of the two methods of approximation, describes this as more accurate and less tedious than any other.

Methods of approximation were also quite numerous for the extraction of the cube root. That of Lucas di Borgo may be seen from the formula,

$$\sqrt[3]{(a^3+x)}=a+\frac{x}{(3a)^3}$$

which Tartaglia says he got from Leonard of Pisa, who had it from the Arabians; and he expresses his surprise that he should have committed so grievous an error, unless he had done so without consideration.

The method of Orontius Fineus is represented by the following formula:

$$\sqrt[3]{(a^3+x)}=a+\frac{x}{3a}$$

which errs as much in excess as that of Di Borgo in defect.

The method of Cardan is indicated by the formula,

$$\sqrt[3]{(a^3+x)}=a+\frac{x}{3a^2}$$

which Tartaglia criticises with great bitterness, as might naturally be expected from one who had been so treacherously defrauded by him of an important discovery, the general method of solving cubic equations. His own method is represented by the formula,

$$\sqrt[3]{(a^3+x)}=a+\frac{x}{3a^2+3a}$$

which, though more accurate than that of Cardan, errs in defect while the other erred in excess.

In later times, methods of approximation have been proposed which give results much more accurate than any of the preceding. One of the very best that we have met is the following, given by Alexander Evans, in the January number of *The Analyst*, 1876:

For square root, $\frac{N}{2r}+\frac{r}{2}$

For cube root, $\frac{N}{3r^2}+\frac{2r}{3}$

For nth root, $\frac{N}{nr^{n-1}}+\frac{n-1}{n}\cdot r$

To illustrate these formulas we will extract the square root of 2 and the cube root of 6. Suppose the square root of 2 is nearly 1.4, then $r=1.4$, and substituting in the formula we have

$$\frac{N}{2r}+\frac{r}{2}=\frac{2}{2(\frac{14}{10})}+\frac{1}{2}\cdot\frac{14}{10}=\frac{99}{70}=1.4142+$$

which is the correct root to four places; and by substituting $\frac{99}{70}$ in the formula we get the root correct to eight places.

In extracting the cube root of 6, suppose that $r=1.8$, then substituting in the formula we have

$$\frac{N}{3r^2}+\frac{2r}{3}=\frac{6}{3(\frac{18}{10})^2}+\frac{2(\frac{18}{10})}{3}=\frac{50}{81}+\frac{6}{5}=1.8172$$

which is true to three decimal places. The method cannot be relied upon, however, to give many correct terms in the approximation. In applying it to the cube root of 3, regarding 1.4 as the value of r, we obtain for the root, 1.44353, which is true to only two places. If we then take 1.44 as the value of r, we shall find the next approximation to be 1.442253, which is true to four places. If we take $r=1.5$ as the cube root of 4, the formula gives the first approximation 1.5925, which is true to only the first decimal place. If we had taken $r=1.6$, we would have obtained 1.5875, which is correct to three places. The best method is therefore the general one; for a person who is familiar with the method which I have given under the name of the *New Method* will extract the root more rapidly than he can with the approximate methods, and may be always certain of the correctness of his result.

PART III.

COMPARISON.

SECTION I.

RATIO AND PROPORTION.

SECTION II.

THE PROGRESSIONS.

SECTION III.

PERCENTAGE.

SECTION IV.

THEORY OF NUMBERS.

SECTION I.

RATIO AND PROPORTION.

I. Introduction.

II. Nature of Ratio.

III. Nature of Proportion

IV. Application of Proportion.

V. Compound Proportion, etc.

VI. History of Proportion.

CHAPTER I.

INTRODUCTION TO COMPARISON.

ARITHMETIC consists fundamentally of three processes; *Synthesis*, *Analysis*, and *Comparison*. Synthesis and Analysis are mechanical processes of uniting and separating numbers; Comparison is the thought process which directs the general processes of synthesis and analysis, and unfolds the various particular processes contained in them. Comparison also gives rise to several processes which do not grow out of the general operations of synthesis and analysis, but which have their origin in the thought process itself The principal processes originating in Comparison, are *Ratio*, *Proportion*, *Progression*, *Percentage*, *Reduction*, and the *Properties of Numbers*. The particular manner in which these processes originate will appear from the following considerations.

If two numbers be compared with each other, we perceive a definite relation existing between them, and the measure of this relation is called *Ratio*. Numbers may be compared in two ways: first, by inquiring how much one number is greater or less than another; and secondly, by inquiring how many times one number equals another. Thus, in comparing 6 with 2, we see that 6 is *four more* than 2, and also that 6 is *three times* 2. These relations, expressed numerically, give us the ratio of the numbers. The former is called *arithmetical ratio;* the latter, *geometrical ratio.* The term *ratio* is generally restricted, however, to a *geometrical ratio*, and it will be thus used here.

The comparison of ratios gives rise to several distinct processes called *Proportion*. If two equal ratios be compared, the numbers producing the ratios being retained in the comparison, we have what we call a *Geometrical Proportion*, or simply a *Proportion*. When the ratios are simple, we have a *Simple Proportion;* when one or both of the ratios are compound, we have a *Compound Proportion.*

If we wish to divide a number into several equal parts, bearing a certain relation to each other, we have a process called *Partitive Proportion.* If we wish to combine numbers in certain definite relations, we have a process called *Medial Proportion*, usually known as *Alligation.* If we compare numbers so that each consequent is of the same kind as the next antecedent, we have a process known as *Conjoined Proportion.*

If we have a series of numbers differing by a common ratio, we may investigate such a series and ascertain its laws and principles; thus arises the subject of *Progressions.* If the ratio is arithmetical, the progression is called an *Arithmetical Progression;* if the ratio is geometrical, the progression is called a *Geometrical Progression.*

Again, as was shown in the Logical Outline of arithmetic, we may take some number as a basis of comparison, and develop the relations of numbers with respect to this basis. It has been found convenient in business transactions to use *one hundred* as such a basis of comparison, which gives rise to the subject of *Percentage.* In Fractions and Denominate Numbers we have units of different values under the same general kind of quantity. By comparing these, it is seen that we may pass from a unit of one value to one of a greater or less value, and thus arises the process of *Reduction.* When we pass from a less to a greater unit the process is called *Reduction Ascending:* when we pass from a greater to a less unit, the process is called *Reduction Descending.*

By a comparison of numbers, we may also discover certain properties and principles which belong to numbers *per se*, and also other properties and principles which have their origin in the Arabic system of notation. Such principles may be embraced under the general head of the *Properties of Numbers.* It is thus seen that several divisions of the science of numbers are not contained in the original processes of synthesis and analysis—that is, of addition and subtraction—but have their roots in and grow out of the thought-process of comparison. These several subjects, evolved from the comparison of numbers, will be considered in the order in which they have been mentioned.

CHAPTER II.

NATURE OF RATIO.

RATIO originates in the comparison of numbers. It is the numerical measure of their relation. From it arise some of the most important parts of arithmetic, as proportion, progressions, etc. Its importance, and the inadequate and diverse views held concerning it, make it necessary to give quite a careful and thorough discussion of the subject.

Definition.—Ratio is the *measure* of the relation of two similar quantities. This definition differs in one respect essentially from that usually given. Ratio is generally defined as "the *relation* of two quantities"—*relation* and *ratio* being made equivalent. This is not accurate, or, at least, not sufficiently definite. The word ratio is a more precise term than relation, as will appear from the following illustration. If we inquire what is the *relation* of 8 to 2, the natural reply is "8 is four times 2;" but if we inquire what is the *ratio* of 8 to 2, the correct reply is "*four.*" Here the ratio *four* is the number which *measures* the relation of 8 compared with 2. It is thus seen that ratio is not merely the *relation* of two similar quantities, but the *measure* of this relation. This definition, presented in the author's own text-books, has already been introduced by one or two writers, and seems not unworthy of general adoption.

The Terms.—A ratio arises from the comparison of two similar quantities. These quantities are called the *terms* of the ratio. The first term is called the *Antecedent*; the second term is called the *Consequent.* The antecedent is compared with the consequent; the consequent is the basis or standard of comparison. Thus, a ratio indicates the value of the first quantity as compared with the second as a standard. The ratio, therefore, expresses how many times the consequent must be taken to produce the antecedent, or what part the antecedent is of the consequent. In other words, it answers the question—the antecedent is *how many times* the consequent, or, the antecedent is *what part of* the consequent? From this it also appears that the ratio equals the *antecedent divided by the consequent.* Thus, the ratio of 6 to 3 is 2, and the ratio of 3 to 6 is $\frac{1}{2}$.

Method of Ratio.—The question has recently been raised whether the correct method of determining a ratio is to divide the antecedent by the consequent or the consequent by the antecedent. An eminent author advocates the division of the consequent by the antecedent, and this method has been adopted by several American mathematicians. The old method some of them call the "English Method;" the new method, the "French Method." The so-called "French Method" we believe to be incorrect in principle and inconvenient in practice. The correct method of finding the ratio of two numbers is to divide the antecedent by the consequent. Several reasons will be given in favor of the correctness of this method, which seem to us conclusive. For convenience in the discussion, let us distinguish the two methods as the *Old* and the *New* method.

1. *Nature of Ratio.*—First, I think the correctness of the Old Method will appear from the nature of ratio itself. If we inquire "What is the relation of 8 to 2?" the natural reply is, "8 is *four* times 2." Here the number *four* is the *measure* of the relation; hence the *ratio* of 8 to 2 is *four.* If the inquiry is, "What is the relation of 2 to 8?" the natural reply is "2 is *one-fourth* of 8;" hence in this case, the ratio is *one-fourth.* From this view of the subject it follows that the correct method of determining a ratio is to divide the antecedent by the consequent, and not the consequent by the antecedent.

If I ask the relation of 8 to 2, it would be illogical to reply, "2 is *one-fourth* of 8," for this does not answer my question. In giving the reply, that number should be used first in making the comparison which was used first in the question, and it would be illogical and absurd to invert the order; yet this is really what those who advocate the other method must do. If the ratio of 8 to 4 is *one-half*, then when I ask the question, "What is the relation of 8 to 4?" they must say, "4 is *one-half* of 8," unless it be supposed that they would say, "8 is *one-half* of 4."

This may be impressed by an illustration suggested by Prof. Dodd. Of two persons, *A* and *B*, suppose *A* to be the *father* and *B* the *son*. Now if the question be asked, "What is the relation of *A* to *B*?" the correct reply is "*A* is the *father* of *B*," and it would be inconsistent to answer, "*B* is the *son* of *A*," for that is the reply to the question, "What is the relation of *B* to *A*?" The same holds in regard to the comparison of numbers, and with even greater force, since it is necessary to be more explicit in science than in ordinary conversation. Hence, if the question is asked, "What is the relation of 8 to 2?" the correct reply is, "8 is *four* times 2;" from which we see that the ratio is *four*. It is clear, then, that the *ratio* of two numbers, which is the *measure* of the *relation* of the *first* to the *second*, is equal to the *first divided by the second*.

2. *Law of Comparison.*—The true method of determining a ratio may also be shown by the nature and object of the comparison. The law of comparison is to compare the *unknown* with the *known;* thus, we logically write $x=4$, and not $4=x$. Now, in a ratio, one number is made the basis of comparison, the object being to comprehend or measure the other number by its relation to the basis. In this sense the basis may be regarded as the known quantity, and the other number as the unknown quantity. Now the unit is the basis of all numbers; it is the standard by which all numbers are measured; we understand a number only as we know its relation to the unit. When any number, as 8, is presented to the mind, we compare

it with the unit, not the unit with it. The inquiry is, 8 is how many times one? hence 8 is the first number named in the comparison; it is, therefore, the antecedent, and the ratio is the quotient of the antecedent by the consequent. The advocates of the new method of ratio would have us compare the 1 with the 8, the unit of measure with the thing to be measured, the known with the unknown. This is not only awkward, but it is directly opposed to the established principles of logical thought.

3. *Authority.*—One of the strongest arguments in favor of the division of the first term by the second is the usage of eminent mathematicians. That signification of scientific terms which custom has fixed should not be changed but for the strongest reasons. From the earliest periods of science, mathematicians have divided the antecedent by the consequent. It was the method employed by Euclid, Pythagoras, and Archimedes, the three great mathematicians of antiquity; and by Newton, LaPlace, and LaGrange, the three great mathematicians of modern times. The English and German, and nearly all the French mathematicians, employ this method, and have done so from the earliest periods. One or two French, and a few American authors have adopted the New Method; but with these few exceptions, the Old Method is the method of mathematicians at all times and in every country where the ratio of numbers has been employed.

But not only is the authority of numbers upon this side of the question, but also the greater weight of the authority of eminence. The practice of all of the great mathematicians of every age is in favor of the Old Method. In its favor we may mention the illustrious names of Euclid, Pythagoras, Archimedes, and to these add the not less illustrious names of Diophantus, Newton, Leibnitz, LaPlace, LaGrange, the Bernoullis, Legendre, Arago, Bourdon, Carnot, Barrow, Herschel, Bowditch, Pierce, etc.; names which shed a lustre over their country and age, and which are symbols of grand achievements in the science. All the great works, the masterpieces which stand as monuments of the loftiest triumphs of genius, are upon this side of the question. The *Principia* of Newton, the *Mécanique Céleste* of LaPlace, the *Mécanique Analytique* of LaGrange, the *Théorie des Nombres* of Legendre, the *Analytical Mechanics* of Pierce, all employ the Old Method. Such universal agreement among great mathematicians should be regarded as a final settlement of the matter.

4. *Inconvenience of the Change.*—Again, the Old Method cannot be changed without confusion. There are definitions in science which involve the idea of ratio, and a correct apprehension of these definitions requires a precise idea of ratio. These definitions are founded upon the Old Method of ratio; hence, if we change the method of determining ratio, we shall either have a wrong idea of the subjects defined, or else the definitions must be changed. The latter would be almost a practical impossibility, since they have become fixed forms in scientific language. Science has embalmed certain definitions, and it would seem almost like sacrilege to disturb them.

Among these definitions may be mentioned those of *specific gravity*, *differential co-efficient*, *index of refraction*, and the geometrical symbol π. The specific gravity of a body is defined to be the *ratio* of its weight to the weight of an equal volume of some other body assumed as a standard. The index of refraction is the *ratio* of the sine of the angle of incidence to the sine of the angle of refraction. The differential co-efficient is the *ratio* of the increment of the function to that of the variable. The geometrical symbol π is the *ratio* of the circumference to the diameter. These definitions have the authority of the great masters, and will, without doubt, remain as they are. One or two of them have been changed by the advocates of the New Method, but such changes will hardly extend beyond their own text-books.

5. *Origin of Symbol.*—It may further be remarked that the assumed origin of the symbol of ratio is in favor of the method

here advocated. It is said that the symbol of ratio is derived from that of division; that is, that : is the symbol ÷ with the horizontal line omitted. The symbol of division indicates that the quantity before it is to be divided by the one following it; hence if the theory of the origin is true, it indicates that primarily the ratio of two numbers was the quotient of the first divided by the second; and this primary method should be followed, unless there are good reasons to the contrary.

In this connection I remark that the Old Method of ratio gives us the simplest idea of a proportion. A proportion is an *equality of ratios*, and this idea is most clearly expressed thus: $6 \div 3 = 8 \div 4$. With the other method of ratio, this simple idea of a proportion cannot be presented. Whether the symbol : is a modification of ÷, is, I presume, not definitely known. It is so asserted by some authors; but so far as I can learn, it is not known as a historical fact. It seems very reasonable, however, and in some old German works I have noticed that the symbol of division is used for indicating the ratio of numbers.

The "French Method," inappropriately so called.—These two methods of ratio have been distinguished by the names "English Method," and "French Method;" the Old Method being called the "English Method," and the New Method the "French Method." These names were first applied, I think, by Prof. Ray, although others had previously stated that the French mathematicians made use of the one and the English mathematicians of the other method. Both of these names are founded in error. The "French Method" is not used by the French; the general custom of the French mathematicians is opposed to it. Lacroix is the only mathematician of any eminence who, so far as I have examined, employs it. The "English Method" is not confined to the English, but it is used by French, Germans, Prussians, and Austrians, in fact, by the mathematicians of all countries, and is, therefore, incorrectly named the English Method.

Nearly all the mathematicians of France, it has been said, employ the so-called English Method, and all of the most eminent ones do so. Among these may be mentioned LaPlace, LaGrange, Legendre, Bourdon, Vernier, Comte, Biot, Carnot, Arago, etc. In proof of this, I will quote from some of their own works. M. Bourdon, in his Arithmetic, page 222, says, "*Par exemple, le rapport de* 24 *à* 6 *est* $\frac{24}{6}$, *ou* 4 *; et celui de* 6 *à* 24 *est* $\frac{6}{24}$, *ou* $\frac{1}{4}$. Legendre, in his Geometry, Book IV., Prop. XIV., says, "*donc le rapport de la circonference au diamètre designé ci-dessus par* π =3.1415926." Vernier, in his Arithmetic, page 118, says, "*comme la raison est le quotient qu' on obtient quand on divise l' antécédent par le conséquent.*" Other authors might be quoted, but these are sufficient to show that the so-called French Method is not the method of the French. Legendre and Bourdon are especially referred to, since some popular American text-books, supposed to be translations from these authors, employ the New Method, and have been instrumental in leading quite a large number of American authors and teachers to adopt that method.

In turning to Lacroix, we see a departure from the general usage of the French mathematicians. In his Arithmetic, which is the only work of his that I have examined, he says, page 85, in comparing the numbers 13, 18, 130, and 180, we see "*que le deuxième contient le premier autant de fois que le quatrième contient le troisième; et ils forment ainsi ce qu'on appelle une proportion.*" Notice that he is here discussing the subject of proportion, and not the subject of ratio by itself. On the next page he remarks, "*Je continuerai de prendre le conséquent du rapport pour le numérateur de la fraction qui exprime le rapport et l' antécédent pour le dénominateur.*"

This places Lacroix upon the opposite side of this question; and it is clear from the manner in which he expresses himself, that he is conscious of taking a position not authorized by the general custom of his countrymen. I think it can readily be seen how Lacroix was led into this error He commences the subject with a problem in proportion, which he solves by analysis, and then, by a mistake plausibly drawn from the process of analysis, seeming to think that the analysis dictates a division of consequent by antecedent, he defines his terms and announces his method of ratio. The whole discussion is as illogical as the conclusion is incorrect. He begins the subject with proportion instead of ratio, thus inverting the whole problem and getting the method of ratio inverted also. The true method is to begin by comparing *numbers*, determining their *relations*; and then comparing their *relations*, make a *proportion*; the first will give the true idea of Ratio, and the second of Proportion.

Answer to Arguments in Favor of the New Method.—This discussion would be imperfect without an attempt to answer some of the arguments which have been presented in favor of the so-called "French Method." An eminent author and educator, who has done more for the adoption of the New Method than any other person in this country, gives a formal defense of it; a few of his arguments I will notice. His first argument, which is founded upon the nature of comparison, has already been answered in the previous discussion. He says, in comparing numbers, "the standard should be the first number named;" hence, to comprehend 8, he would compare the basis of numbers, or 1, with 8, instead of comparing the 8 with 1, that is, the number with the basis. The mistake he makes is in comparing the standard with the thing measured; that is, the known with the unknown; the true law of comparison being just the reverse of this.

This will be readily seen in continuous quantity which can be clearly understood only by comparing it with some definite part of itself assumed as a unit. Thus, suppose a period of time is considered; it is clear that we can get a definite idea of it by comparing it with some fixed unit, as a *day*, or a *week*, or a *year*. In these cases it will be seen that we do not compare the unit with the given quantity, as the author quoted would maintain, but the quantity to be measured with the unit of measure.

His second argument is that the New Method gives a convenient rule for Proportion; the fourth term being equal to the third term *multiplied* by the ratio of the first to the second. The reply is that the Old Method gives just as convenient a rule, namely, "The fourth term equals the third *divided* by the ratio of the first to the second." His third argument is, that in a geometrical progression the ratio is the quotient of any term divided into the following term. This is the most plausible argument advanced, and demands special notice. If it be true that the ratio of any term to the following term is the quotient of the second divided by the first, then it is true that we here depart from the general method of ratio; but still it would not follow that the general method of ratio should be changed to harmonize with this exceptional case. A more sensible conclusion would be that the method here used should be changed to correspond with the general method. That the general should control the special and not the special the general, is a fixed law of science. Let us see, however, if the form of writing a geometrical progression does present an exception to the general method of expressing a ratio.

In a geometrical progression, the *ratio* is the measure of the relation that any term bears to the preceding term. In the series 1, 2, 4, 8, etc., we do not compare the 1 with the 2, the 2 with the 4, etc., to determine the ratio, as will appear from the following considerations. Suppose, for illustration, that we wish to find any term of the series, as the 5th term, would we not reason thus: the 5th term must bear the same relation to the 4th, that the 4th does to the 3d; and since the 4th is twice the 3d, the 5th term must be twice the 4th, or 16. Here we follow the law of comparison, by comparing the *unknown* with the *known*, and reversing the apparent order, name the 8 first and the 4 after it. Should we write the comparison out in full, we would have 5th : 8 : : 8 : 4. If this is true, then, in a geometrical series, we do not compare a term with the following term, but rather with the term preceding it. The ratio of the series, it thus appears, is the ratio of any term to the preceding term, and not to the term following it. In other words, we compare backward, instead of forward, as in ordinary ratio; and really divide the antecedent of the comparison by the consequent to obtain the ratio.

Some writers explain this apparent departure from the general signification of ratio, by saying that in a geometrical series we express the "inverse ratio of the terms." Says one, "It is less troublesome to express the common ratio inversely, as then one number will suffice." Says another, "Whenever we meet with the expression, the 'ratio of a geometrical series,' we are to understand the inverse ratio." It seems clearer to me to say that the order of writing the terms is in opposition to the order of thought. We write one way and compare another way. If the expression of the series were dictated by the idea of ratio, we would write it from the right toward the left.

The fact is, however, that in a geometrical progression, it is the *rate* of the progression that we consider, rather than the ratio of the terms; that is, the rate at which the series progresses, and this term would be preferable to *ratio* in this connection. A series of terms, increasing or decreasing by a common multiplier, although an outgrowth from the idea of ratio, presents an idea not identical with that of ratio.

This distinction is actually made by several French writers. They use the different words *rapport* and *raison;* the former to express the ratio of two numbers, the latter to denote the *rate* of the geometrical series. Thus Bourdon, in his Arithmetic, page 279, says, "*On appelle Progression par Quotient une suite de nombres tels que le rapport d'un terme quelconque à celui qui le précède est constant dans toute l'étêndue de la série. Ce rapport constant, qui existe entre un terme et celui qui le précède immédiatement se nomme la Raison de la progression.*" Prof. Henkle, who has written several excellent articles upon this subject, quotes Biot to the same effect. He says of a geometrical progression, "*Le Rapport de chaque terme au precedent se* *nomme Raison.*" It will thus be seen that some of the French writers distinguish between *ratio* and the constant multiplier of a progression, and should the word *rate* be adopted with us, we would avoid the objection of this seeming departure from the general signification of ratio.

I have devoted so much space to the discussion of this subject, because I think it one upon which there should be uniformity of opinion and practice. Several of our most popular elementary text-books on mathematics have adopted the so-called "French Method," and are teaching it to the youth of the country. Pupils who have been taught the method can with difficulty relinquish it, and if they proceed to Philosophy and Higher Mathematics they will meet with difficulty in every subject containing definitions involving ratio. It is proper to remark that since this article was written, now some ten or twelve years, several authors who had adopted the new method, have discarded it and now use the old method.

CHAPTER III.

NATURE OF PROPORTION.

PROPORTION arises from the comparison of ratios. Comparison begins with comparing numbers, giving rise to the idea of relation, the measure of which is ratio. After becoming familiar with the idea of the relations of numbers, we begin to compare these relations; when equal relations are compared, we attain to the idea of a Proportion.

Proportion, it is thus seen, has its origin in comparison; it is a comparison of the results of two previous comparisons. Every proportion involves three comparisons; the two which give rise to the ratios, and a third, which compares or equates the ratios. All of these comparisons are exhibited in the expression of a proportion; the symbol of ratio in the two couplets showing the first two, and the symbol of equality between the couplets showing the third. A proportion, therefore, involves four numbers, so arranged that it will appear that the ratio of the first to the second equals the ratio of the third to the fourth. Thus, the ratio of 6 to 3 being the same as the ratio of 8 to 4, if they are formally compared, as $6 : 3 = 8 : 4$, we have a proportion.

Notation.—A proportion may be written by placing the sign of equality between the two ratios compared; thus $2 : 4 = 3 : 6$. Instead of the sign of equality, the double colon is generally used to express the equality of ratios, the proportion being written, $2 : 4 :: 3 : 6$. The symbol of equality, however, is frequently used by the French and German mathematicians, and is always to be preferred in presenting the subject to learners. A proportion may be read in several different ways. Thus we may read the above proportion,—"the ratio of 2 to 4 equals the ratio of 3 to 6;" or "2 is to 4 as 3 is to 6." The latter is the method generally used.

Definition.—A Proportion is the comparison of two equal ratios; or, it is the expression of the equality of equal ratios. In this expression the numbers that are compared to obtain the ratio must be indicated. A proportion is thus seen to be an equation, and should be thus regarded. An equation, as generally used, expresses the relation of equal numbers; a proportion expresses the relation of equal ratios One arises from the comparison of quantities; the other, from the comparison of the relations of quantities. The former is an equation between equal numbers; the latter is an equation between equal ratios.

The definition of proportion generally given is, "A proportion is an equality of ratios." This is true, but it is not sufficiently definite to constitute a perfect definition. There must be not only an equality of ratios, but a formal comparison of these ratios, to produce a proportion. This comparison must also exhibit the numbers which were compared to produce the equal ratios. Thus, the ratio of 6 to 3 is 2, and the ratio of 8 to 4 is 2; here is an equality of ratios, but not a proportion. Again, if we compare the ratios 2 and 2, we have the equation $2=2$, which is not a proportion, since it does not exhibit the numbers which produce the equal ratios. To give a proportion, it is essential that the ratios be compared, and that the comparison of the numbers which give the ratios be exhibited. The mere equating of the ratios is not sufficient; the proportion must show the numbers which, compared, give rise to the equal ratios. A proportion, then, is not only an "equality of ratios," but it is a comparison of equal ratios, in which the comparison of the numbers compared for a ratio is exhibited.

This idea of the exhibition of the numbers compared for the ratios, though not formally stated in the definition which I have presented, may be directly inferred from it. For, if we compare as above, 2=2, so far as we can see, it is merely a comparison of numbers, and not a comparison of ratios. It is true that every ratio is a number, but the converse is not true; hence 2=2 may or may not be the comparison of two ratios. Such comparison would be indefinite; therefore, to express definitely and clearly the equality of ratios, we must retain the numbers compared, to show that the equation is an expression of equal ratios, and not a mere comparison of numbers. The definition is consequently regarded as sufficiently explicit to prevent any misapprehension. Should we wish to incorporate this idea in the definition, we might define as follows: A Proportion is a comparison of equal ratios, in which the numbers producing the ratios are exhibited.

Kinds of Proportion.—There are several kinds of proportion, resulting from a modification or extension of the primary ideas of ratio and proportion. A comparison of three or more pairs of numbers having equal ratios, is called *Continued Proportion.* An expression of the equality of compound ratios is called *Compound Proportion.* An *Inverse Proportion* is one in which two quantities are to each other inversely as two other quantities. An *Harmonical Proportion* is one in which the first term is to the last as the difference between the first and second is to the difference between the last and the one preceding the last. We have also *Partitive* and *Medial Proportion,* which will be defined subsequently. The proportion requiring special consideration is *Simple Proportion,* or the comparison of two simple ratios.

Principles.—The principles of Proportion are the truths which belong to it, and which exhibit the relations between the different members. The fundamental principle of Proportion is that *the product of the means equals the product of the extremes.* From this we derive several other principles by which we can find the value of either of the four terms when the other three are given. There are many other beautiful principles of Proportion, besides this fundamental one and its immediate derivatives, which are not usually presented in arithmetic, but may be found in works on algebra and geometry. They are, however, just as much an essential part of pure arithmetic as of geometry, and can all be demonstrated as easily here as there. Indeed, they belong to arithmetic rather than to geometry, since a ratio is essentially numerical, and hence should be treated in the science of numbers. These principles, it will be seen, are not self-evident; they admit of demonstration. Remembering this, it may be asked, what then becomes of the assertion of the metaphysicians, that there is no reasoning in pure arithmetic?

Demonstration.—The fundamental principle of Proportion may be demonstrated in two ways. The method generally given is the following: Take the proportion 4 : 2 :: 6 : 3. From this we have $\frac{4}{2}=\frac{6}{3}$; clearing of fractions, we have $4\times 3 = 2\times 6$; and, since 4 and 3 are the extremes, and 2 and 6 the means, we infer that *the product of the extremes equals the product of the means.* This is the method generally used in algebra and geometry. Although entirely satisfactory as a demonstration, the objection might be made that though it *proves* that the products are equal, it does not show *why* they are equal.

Another method which, in arithmetic, is preferred to the above, is as follows: From the fundamental idea of ratio and proportion, we see that in every proportion we have 2d term × ratio . 2d term :: 4th term × ratio : 4th term. Now, in the product of the extremes, we have 2d term, ratio, and 4th term, and in the product of the means, we have the same factors; hence the products are equal. This is a simple method, clearly seen, and shows not only that the products *are* equal, but that they *must* be so, and *why* they are so, which the other method does not. The products are seen to be equal because in the very nature of the subject they contain the same factors.

The same demonstration may be put in the more concise language of algebra. Take the proportion $a : b :: c : d$, let r = the ratio, then we have $a \div b = r$, hence $a = b.r$, and in the same way $c = d.r$; hence the proportion becomes $b.r : b :: d\,r : d$. Now, in the extremes we have b, r, and d, and in the means we have the same factors; hence the two products will be equal.

CHAPTER IV.

APPLICATION OF SIMPLE PROPORTION.

SIMPLE PROPORTION is employed in the solution of problems in which three of four quantities are given, to find the fourth. These quantities must be so related that the required quantity bears the same relation to the given quantity of the same kind that one of the two remaining quantities does to the other. We can then form a proportion in which one term is unknown, and this unknown term can be found by the principles of proportion. Thus, suppose the problem to be,—What cost 3 yards of cloth, if 2 yards cost $8?

OPERATION.

Cost of 3 yds. : \$8 :: 3 yds. : 2 yds;

Cost of 3 yds. $=\frac{8\times 3}{2}=$ \$12.

Here we see that the *cost of* 3 *yards* bears the same relation to the *cost of* 2 *yards* that 3 *yards* bears to 2 *yards*; hence we have the proportion given in the margin, from which we readily find the value of the unknown term.

In all such problems three terms are given to find the fourth; from which Simple Proportion has been called the ***Rule of Three.*** It was regarded as very important by the old school of arithmeticians, and was by them called "The golden rule of three." It is now falling into disrepute, the beautiful system of analysis having, to a great extent, taken its place. The method of analysis is simpler in thought than that of proportion, and in many cases is to be preferred to the solution by proportion, especially in elementary arithmetic; but still the rule of

Simple Proportion should not be entirely discarded. The comparison of elements by proportion affords a valuable discipline and should be retained for educational reasons; and moreover it is also valuable, if not indispensable, in the solution of some problems which can hardly be reached by analysis. In algebra, geometry, and the higher mathematics, it is, of course, indispensable.

Position of the Unknown Quantity.—It is seen that, in the solution of the preceding problem by proportion, I place the unknown quantity in the first term. This is not in accordance with general custom; other writers place the unknown quantity in the fourth term. I have ventured to depart from this custom, and to recommend the general adoption of such a departure, for reasons which seem to me conclusive. These reasons are twofold: first, the method suggested is dictated by the laws of logic; and, second, it is more convenient in practice. Both of these points will be briefly considered.

First. The law of correct reasoning is to compare the *unknown* with the *known*, not the *known* with the *unknown*. The ordinary method begins the proportion with the known quantities, thus comparing the known with the unknown, in violation of an established principle of logic. The method I have suggested commences with the unknown quantity, and thus compares the *unknown* with the *known*, in conformity to the laws of thought. It seems therefore that the old method is not logically accurate, and that the correct method of solving a problem in Rule of Three is to place the unknown quantity in the first term.

Second. The method proposed will be found to be much more convenient in practice. A proportion is more easily stated by beginning it with the unknown term. This will be especially appreciated by those who have taught Trigonometry. In stating a proportion so as to get the required quantity in the last term, I have seen pupils try two or three statements before obtaining the right one. It cannot be readily seen how the proportion should begin so that the unknown quantity shall come in the last term. If, however, the pupil begins the proportion with that which he wishes to find, the other terms will arrange themselves without any difficulty. Suppose, for instance, that we wish to obtain an unknown angle of a triangle. If we reason thus: sine of the required angle is to the sine of the given angle as the side opposite the required angle is to the side opposite the given angle; the pupil will write the proportion without any hesitation. If we reverse this order, it is necessary to go through the whole comparison mentally before beginning to write, so that we may be sure to close the proportion with the required quantity. It is therefore believed that the simplest method of stating a proportion is to place the unknown quantity in the first term.

The utility of this change has been frequently illustrated in my own experience. I remember, while visiting a young women's college, hearing a recitation in geometry in which the professor was trying to lead a pupil to state a proportion from which a certain line could be determined. The young lady made several attempts and failed, when I said, "Professor, let her begin with the line she wishes to find." He accepted the suggestion, and she immediately stated the proportion correctly.

Several authors' suggest that the unknown quantity should be placed sometimes in one term and sometimes in another to test the pupil's knowledge of the subject. This is a valuable suggestion; but any position of the unknown term except in the fourth term they regard not as a general, but as an exceptional method. Their rule is to place the unknown term last; any other arrangement is the exception. What I claim is that the placing of the unknown quantity in the first term should be the rule, and any other arrangement the exception. It is recommended also that the teacher require the learner to place it in different terms, that he may acquire a clear and complete idea of the subject.

Symbol for the Unknown.—Some authors employ the letter x in arithmetic as a symbol for the unknown quantity. Thus, in the problem previously presented, we may write $x : \$8 :: 3 : 2$. This practice is derived from the French, and is commendable. It is sometimes objected, that it is introducing algebra into arithmetic; but such objection, however, is not valid. Algebra and arithmetic are not two distinct sciences, but rather branches of the same science. The former, at least in its elements, is but a more general kind of arithmetic; and it is not at all improper to introduce its concise and general language into arithmetic. I think it well, with younger pupils, to express the unknown term in an abbreviated form as is indicated in the previous solution; when pupils become familiar with this, I would use the symbol x as a representative of it.

Three Terms Statement.—It is seen that in the solution of the given problem in proportion, I use four terms in the statement. Many authors, however, use only three terms in stating a proportion. This was the method of the old authors, when rules reigned and principles were ignored, in what might be called "the dark ages" of arithmetic. Several recent writers have broken away from the old usage, and write the proportion with four terms instead of three. It is unnecessary to say that the old method was incomplete and incorrect. An expression is not a proportion unless it has four terms. The old method was merely mechanical, and gave the pupil no idea, or at least a very imperfect idea, of the true nature of proportion. The sooner the new method is generally adopted the better for science and education.

Method of Statement.—No subject in arithmetic is so illogically presented as Simple Proportion in its application to the solution of problems. In the statement of the proportion, all reasoning seems to be completely ignored, and the whole thing becomes a mere mechanical operation for the answer. The process is as follows: "Write that number which is like the answer sought as the third term; then if the answer is to be greater than the third term, make the greater of the two remaining numbers the second term and the smaller the first term," etc.

Now, though this might do well enough as a rule for getting an answer, to require the pupils to explain the solution by it, as is done in many instances, is to rob the subject of any claims to a scientific process. The pupil thus taught to solve his problems has no more idea of proportion than if the subject were not presented in the book. The whole process becomes a piece of charlatanism, utterly devoid of all claims to science. A better rule would be this: *Write the number like the answer; if the answer is to be greater, multiply by the greater of the other two numbers and divide by the less, etc.* This would be the better method, since it makes no claims to be a scientific process, as the other does. Both methods are absurd as a process of reasoning in Arithmetic; but the latter less so, since it makes no pretensions to be a reasoning process.

What then is the true method? I answer, if a pupil cannot state a proportion by actual comparison of the elements of the problem, he is not prepared for proportion, and should solve the question by analysis. If he uses proportion, he should use it as a logical process of reasoning, and not as a blind mechanical form to get the answer. He should then be required to reason thus: Since the cost of 3 yds. bears the same relation to the cost of 2 yds. that 3 yds. bear to 2 yds., we have the proportion, cost of 3 yds. : $8 : : 3 yds. : 2 yds.

If this is not evident and cannot be readily seen, then we should dispense with proportion until the pupil is old enough to understand it, and require the problems to be solved by analysis. If the unknown quantity be placed in the last term we would reason thus: Since 2 yds. bear the same relation to 3 yds. that the cost of 2 yds. bears to the cost of 3 yds, we have the proportion, 2 yds. : 3 yds. : : $8 : cost of 3 yds.

Cause and Effect.—A new method of explaining proportion has recently been introduced into arithmetic, which may be called the method of *Cause and Effect.* All problems in proportion, it is said, may be considered as a comparison of *two causes* and *two effects;* and since effects are proportional to causes, a problem is supposed to be readily stated in a proportion. To illustrate, take the problem, *If 2 horses eat 6 tons of hay in a year, how much will 3 horses eat in the same time?* Here the *horses* are regarded as a *cause* and the *tons of hay* as an *effect*, and the reasoning is as follows: 2 horses as a *cause* bear the same relation to 3 horses as a *cause*, that 6 tons as an *effect*, bears to the required *effect*; from which we have a proportion and can determine the required term.

This method was first introduced into arithmetic by Prof. H. N. Robinson, and has been adopted by several authors. The same idea was presented by an arithmetician of Verona, who distinguished the quantities into *agents* and *patients*. It is supposed that it tends to simplify the subject, enabling learners more readily to state a proportion than by a simple comparison of the elements. This supposition, however, is not founded in truth. Instead of simplifying the subject, the method of cause and effect really increases the difficulty and tends to confuse the mind. It lugs into arithmetic an idea foreign to the subject, to explain relations which are much more evident than the relation of cause and effect.

Another objection to the method is that the relation of quantities as cause and effect is often rather fancied than real. In many cases, indeed, there is no such relation existing at all. Take the problem, "If a man walks 6 miles in 2 hours, how far will he walk in 5 hours?" Will the pupil readily see which is the cause and which the effect? Will the advocates of the method, tell us whether the 6 *miles* or the 2 *hours* are to be regarded as the cause? Or take the problem, "If 18d. sterling equal 36 cts. U. S., what are 54d. sterling worth?" Would not the pupils be puzzled to tell which is the cause and which the effect? Indeed, there is no relation of cause and effect in a large number of such problems; and any effort to establish such a relation will confuse that which is simple and easily understood.

If anything further is needed to show the incorrectness of the method, take a problem in what is called Inverse Proportion. Thus, "If 3 men do a piece of work in 8 days, in what time will 6 men do it?" Here 3 *men* and 8 *days* would be regarded as the first cause and effect, and 6 men and the corresponding number of days as the second cause and effect. Now, if we form a proportion, we have the *first cause* is to *second cause* as the *second effect* is to the *first effect;* from which we see that in this case *like causes are not to each other as like effects*, a conclusion which completely contradicts the fundamental principle of the relation of cause and effect.

Inverse Proportion.—There is a class of problems which give rise to what is called Inverse Proportion. In this the two quantities of the same kind are to each other, not directly as the other two quantities in the order of their relation, but rather inversely as those quantities. Thus, in the problem, "If 3 men build a fence in 12 days, in what time will 9 men build it?" Here we have the required time is to 12 days, not as 9 men to 3 men, but as 3 men to 9 men; that is, inversely as the order indicated by the order of the terms of the first couplet. This is sometimes called Reciprocal Proportion, since the quantities are as the reciprocals of 9 and 3; that is as $\frac{1}{9}$ to $\frac{1}{3}$ or 3 to 9

Many problems in Inverse Proportion may, however, be stated in a direct proportion. To illustrate, take the problem just solved. Now, if 3 men do a piece of work in 12 days, in 1 day they will do $\frac{1}{12}$ of it, and if a number of men do a piece of work in 4 days, in 1 day they will do $\frac{1}{4}$ of it; hence, since the number of men are to each other as the work done, we have the direct proportion, "the number of men required is to 3 men, as $\frac{1}{4}$ to $\frac{1}{12}$," from which we can readily find the term required. If, in this proportion, we multiply the second couplet by 48, it will become 12 : 4, which gives the same proportion as that which was obtained by the method of inverse proportion. It is thus seen that, in some cases at least, the method of inverse proportion may be avoided, and the problem be expressed by a direct proportion.

If, however, in the above problem the number of men in both cases had been given, and the number of days in one case required, the problem could not be conveniently stated in a direct proportion, since to do so would require the reciprocal of the unknown quantity. Should this quantity be represented by an algebraic symbol, however, we could still state the proportion directly, and readily find the unknown quantity.

Proportion distinctly Arithmetical.—The subject of proportion is purely an arithmetical process. Ratio is a *number*, hence proportion, arising from the comparison of ratios, must be numerical. These ratios may arise from comparing continuous or discrete quantities, hence we may have a proportion wherein geometrical quantities are compared. Attention is called to the fact, however, that the principles of proportion are only generally true with respect of numbers. A proportion in geometry, comparing four surfaces or volumes, may be true, but the principles of a proportion can have no meaning in such a case. In taking the product of the means equal to the product of the extremes, we shall have one surface or one volume multiplied by another, which can mean nothing unless they be regarded as numbers. In geometry we regard the product of two lines as giving a surface, and the product of a line and surface as giving a volume; but what idea can we attach to the product of two surfaces or two volumes? It is thus seen that Proportion is essentially a process of numbers, and is, therefore, a branch of Pure Arithmetic. Since the principles of Proportion admit of demonstration, we inquire again what becomes of Mansel's assertion that "Pure Arithmetic contains no demonstration?"

CHAPTER V.

COMPOUND PROPORTION.

A COMPOUND PROPORTION is a proportion in which one or both ratios are compound. It is employed in the solution of problems in which the required term depends upon the comparison of more than two elements. In Simple Proportion the unknown quantity depends upon a comparison of two elements forming one pair of similar quantities; in Compound Proportion it depends upon the comparison of several elements forming two or more pairs of similar quantities.

A Compound Ratio has been defined as the product of two or more simple ratios. The expression of a compound ratio is $\left\{\begin{matrix} 2 : 4 \\ 5 : 10 \end{matrix}\right\}$. If such a ratio be compared to an equal simple ratio, or if two such compound ratios be compared with each other, we have a compound proportion. Thus $\left\{\begin{matrix} 3 : 6 \\ 2 : 8 \end{matrix}\right\} :: 7 : 56$ and $\left\{\begin{matrix} 2 : 4 \\ 5 : 10 \end{matrix}\right\} :: \left\{\begin{matrix} 3 : 6 \\ 7 : 14 \end{matrix}\right\}$ are examples of compound proportion. In these expressions we mean that the value of the first couplet equals the value of the second; thus, in the first proportion we have $\frac{3}{6} \times \frac{2}{8}$ or $\frac{6}{48}$ equals $\frac{7}{56}$; in the second, $\frac{2}{4} \times \frac{5}{10} = \frac{3}{6} \times \frac{7}{14}$.

The subject of Compound Proportion has been even more unscientifically treated, if possible, than Simple Proportion. In no work upon Arithmetic, and indeed in no work upon Algebra, have I seen the subject presented in a really scientific manner. As a general thing, problems are given under the head of compound proportion, to be solved either mechanically by rule, or else by analysis, which, of course, is not compound proportion.

The principles of a compound proportion are not developed, and in its application it is regarded, not as a scientific process, but as a machine for working out the answer. This, of course, is not as it should be. Compound Proportion is just as much a scientific process as Simple Proportion, and demands just as logical a treatment. I will enforce what I mean by calling attention to a few of the principles of such a proportion, and then showing its scientific application.

Principles.—In Compound Proportion we have certain definite scientific principles, as in Simple Proportion. A few of these principles will now be stated.

1. *The product of all the terms in the means equals the product of all the terms in the extremes.* To show the truth of this, take the proportion given in the margin. From the principles of compound ratio we have $\frac{2}{4}\times\frac{5}{10}=\frac{3}{6}\times\frac{7}{14}$; and clearing this of fractions we have $2\times 5\times 6\times 14=3\times 7\times 4\times 10$, which, by examining the terms, we see proves the principle. From this principle we can immediately derive two others.

OPERATION.

$$\left\{\begin{matrix}2 : 4\\5 : 10\end{matrix}\right\} :: \left\{\begin{matrix}3 : 6\\7 : 14\end{matrix}\right\}$$

$$\tfrac{2}{4}\times\tfrac{5}{10}=\tfrac{3}{6}\times\tfrac{7}{14}$$

$$2\times5\times6\times14=3\times7\times4\times10.$$

2. *Any term in either extreme equals the product of the means, divided by the product of the other terms in the extremes.*

3. *Any term in either mean equals the product of the extremes divided by the product of the other terms in the means.*

Other principles can also be derived, as in Simple Proportion, but the three given are all that are necessary in arithmetic.

Application.—In the application of Compound Proportion to the solution of problems, we should proceed upon the same principles of comparison employed in Simple Proportion. If we do not, the process is not Compound Proportion, and should not be so regarded. To illustrate the true method, we take the problem, "If 4 men earn $24 in 7 days, how much can 14 men earn in 12 days?"

In the solution of this problem by Compound Proportion, we should reason thus: The sum earned is in proportion to the number of men and the time they labor; hence the sum 14 men can earn is to $24, the sum that 4 men earn, as 14 men to 4 men, and also as 12 days to 7 days; giving the compound proportion which is presented in the margin. From this we find the unknown term to be $144. Or we may enter a little more into detail, and say—The sum 14 men can earn in 7 days is to the sum 4 men can earn in 7 days, as 14 men is to 4 men; and also the sum 14 men can earn in 12 days is to the sum that they can earn in 7 days, as 12 is to 7; hence we have the compound proportion given in the margin.

OPERATION.

$$\text{Sum} : 24 :: \begin{cases} 14 : 4 \\ 12 : 7 \end{cases}$$

$$\text{Sum} = \frac{24 \times 14 \times 12}{4 \times 7}$$

By Analysis.—The subject of Compound Proportion is somewhat difficult, in fact too difficult, for young students in arithmetic. With such the method of analysis should be used instead of proportion. The analytical method is clear and simple, and will be readily understood. It should be borne in mind, however, that when we solve by analysis we are not solving by compound proportion, a fact that seems sometimes to be forgotten.

In solving the preceding problems by analysis, it is necessary to pass from the 4 men to 14 men, and from the 7 days to 12 days, the sum earned varying as we make the transposition: to do this we pass from the collection to the unit, and then from the unit to the collection. The solution is as follows, the work being as indicated in the margin.

If 4 men earn $24 in 7 days *one* man will earn $\frac{1}{4}$ of $24, and 14 men will earn 14 times $\frac{1}{4}$ or $\frac{14}{4}$ of $24. If 14 men earn $\frac{14}{4} \times \$24$ in 7 days, in *one* day they will earn $\frac{1}{7}$ of $\frac{14}{4}$ of $24, and in 12 days they will earn 12 times $\frac{1}{7}$ of $\frac{14}{4}$ of $24, which is $\frac{12}{7}$ of $\frac{14}{4}$ of $24, which by cancelling, we find equals $144. Instead of putting it in the form of a compound fraction, we could

OPERATION.

$$\text{Sum} = \tfrac{12}{7} \times \tfrac{14}{4} \times \$24.$$

have made the reduction as we passed along; but in complicated problems the method here used is preferred, as the cancellation of equal factors will often greatly abridge the process.

PARTITIVE PROPORTION.

The subject of ratio gives rise to several arithmetical processes which have received the name of Proportion. Among these we have Partitive Proportion, Conjoined Proportion, Medial Proportion, Geometrical Proportion, etc. Geometrical Proportion embraces Simple Proportion, Compound Proportion, Inverse Proportion, etc. The other kinds are distinguished by their special names. When we speak of proportion, without any qualifying word, we mean Geometrical Proportion. Geometrical Proportion has been treated in the preceding part of this chapter; the other varieties of proportion will now be presented.

The comparison of numbers gives rise to a division of them into parts which shall bear a *given relation* to each other. This process has received the name of Partitive Proportion. Partitive Proportion is the process of dividing numbers into parts bearing certain relations to each other. To illustrate, suppose it be required to divide 24 into two parts, one of which is twice the other. An equivalent problem is, "Given the sum of two numbers equal to 24, and one of the numbers twice the other; what are the numbers?"

Origin.—Partitive Proportion is a process of pure arithmetic; it originated, however, in the application of numbers to business transactions. Partnership is a case of Partitive Proportion. But, although the subject had its origin in the application of numbers, it is now, in accordance with the law of the growth of science, a purely abstract process.

Cases.—This subject embraces quite a large number of cases, arising from the various relations that may exist among the several parts into which a number is divided. It is evident, also, that the greater the number of the parts the more complicated will become the process. The most important cases are the following:

1. When the parts are all equal.
2. When one part is a number more or less than the other.
3. When one part is a number of times the other.
4. When one part is a fractional part of the other.
5. When the parts are to each other as given integers.
6. When the parts are to each other as given fractions.
7. When a number of times one part equals a number of times another.
8. When a fractional part of one equals a fractional part of another.

These simple cases, it is evident, may be combined with each other, giving rise to others more complicated than any of these. A little ingenuity will suggest a large number of such cases, some of which will be quite interesting.

Method of Treatment.—To illustrate the character of one of the simple cases and its treatment, let us take a problem and its solution. Case 8 will give us a problem like the following: Divide 34 into two parts such that $\frac{2}{3}$ of the first part equals $\frac{3}{4}$ of the second part. The solution of this case is as follows: If $\frac{2}{3}$ of the first equals $\frac{3}{4}$ of the second, $\frac{1}{3}$ of the first equals $\frac{1}{2}$ of $\frac{3}{4}$ or $\frac{3}{8}$ of the second, and $\frac{3}{3}$ of the first equals $\frac{9}{8}$ of the second; then $\frac{9}{8}$ of the second, which is the first, plus $\frac{8}{8}$ of the second, or $\frac{17}{8}$ of the second part, equals 34, etc. The other cases are solved in my Mental Arithmetic, and need not be presented here.

CONJOINED PROPORTION.

The comparison of numbers also gives rise to an arithmetical process which has received the name of Conjoined Proportion. Conjoined Proportion is the process of comparing terms so related that each consequent is of the same kind as the next antecedent. The character of the subject is seen by the following concrete problem: "What cost 8 apples, if 4 apples are worth 2 oranges, and 3 oranges are worth 6 melons, and 4 melons are worth 12 cents?"

An abstract problem, showing that it is a process of pure arithmetic, is as follows: "If twice a number equals 4 times another number, and 3 times the second number equals 6 times a third number, and 4 times the third number equals 2 times a fourth number, and 5 times the fourth number equals 40; what is the first number?"

Method of Treatment.—Conjoined Proportion is treated by analysis, and presents a very interesting application of the analytical method of reasoning. The problems may be solved in two ways somewhat distinct; that is, we may begin at the latter part of the problem, and work back, step by step, to the beginning; or we may commence at the beginning of the problem and pass from quantity to quantity, in regular order, until we find the value of the first quantity in terms of the last. To illustrate, the problem given may be solved thus:

Solution 1.—If 5 times the fourth number equals 40, once the fourth number equals $\frac{1}{5}$ of 40, or 8, and twice the 4th, which equals 4 times the 3d, equals 2 times 8, or 16. If 4 times the 3d equals 16, *once* the 3d equals $\frac{1}{4}$ of 16, or 4, and 6 times the 3d or 3 times the 2d equals 6 times 4, or 24; and so on until we reach once the 1st number.

Solution 2.—If *twice* a number equals 4 times another, once the number equals $\frac{1}{2}$ of 4 times, or two times the 2d; if 3 times the 2d equals 6 times the 3d, *once* the 2d equals $\frac{1}{3}$ of 6 times, or 2 times the 3d, and 2 times the 2d, or the 1st, equals *twice* 2 times the 3d, or 4 times the 3d; and so on until we find once the 1st in terms of the given quantity.

Both of these methods are simple and logical. The first method will probably be preferred for its directness and simplicity. It may also be remarked that these problems can be solved by Compound Proportion, and perhaps might have been logically treated under that head.

MEDIAL PROPORTION.

The comparison of numbers and the combining of them in certain relations, give rise to an arithmetical process which has received the name of Medial Proportion. Medial Proportion is the process of finding in what ratio two or more quantities may be combined, that the combination may have a mean or average value.

The subject, in its application, is usually called Alligation, from *alligo*, I bind or unite together, the name being suggested, probably, by the method of solution, which consisted of linking or uniting the figures with a line. It may, however, have been suggested by the nature of the process itself, in which the several quantities are combined.

Origin.—Medial Proportion also originated in the concrete, that is, in the application of numbers. Indeed, even now it is difficult to present it as an abstract process; that is, as a process of pure number. It is so intimately associated with the combination of things of different values, that it is very difficult to apply it to the combination of abstract numbers. Still it is evidently a process of pure arithmetic; and its importance and distinctive character, even as an application of numbers, lead me to speak of it in this connection.

Cases.—The subject presents a number of cases, the most important of which are the following:

1. Given, the quantity and value of each, to find the mean value.

2. Given, the mean value and the value of each quantity, to find the proportional quantity of each.

3. Given, the mean value, the value of each, and the relative amounts of two or more, to find the other quantities.

4. Given the mean value, the value of each, and the quantity of one or more, to find the other quantities.

5. Given, the mean value, the value of each, and the entire quantity, to find the quantity of each.

Method of Treatment.—As formerly treated, the subject was one of the most mechanical in arithmetic. The old "linking process," as presented in the text-books, was seldom understood either by teacher or pupil. Recently, however, Prof. Wood, formerly of the New York State Normal School, has made a very happy application of analysis to the solution of this class of problems, and poured a flood of light upon the subject, so that it is now one of the most interesting processes of arithmetic. It has extended the domain of the subject also, so that it includes some of the more difficult cases of Indeterminate Analysis, for an illustration of which see my Higher Arithmetic.

The method of treatment is to compare one number above the average with one below it by their relation to the average, finding how much must be taken to gain or lose a unit on the one and balancing it with the loss or gain of a unit on the other. In this way the quantities are balanced around the average, and the proportional parts of the combination derived. For an illustration of the method of treatment, see my written arithmetics.

CHAPTER VI.

HISTORY OF PROPORTION.

THE Rule of Three, emphatically called the Golden Rule, by both ancient and modern writers on arithmetic, is found in the earliest writings upon the science of numbers. In the *Lilawati* the rule is divided, as among modern writers, into direct and inverse, simple and compound, with statements for performing the requisite operations, which are said to be quite clear and definite.

The terms of the proportion in the *Lilawati* are written consecutively, without any marks of separation between them. The first term is called the *measure* or *argument;* the second is its *fruit* or *produce;* the third, which is of the same species as the first, is the *demand, requisition, desire,* or *question*. When the *fruit* increases with the increase of the *requisition*, as in the direct rule, the second and third terms must be multiplied together and divided by the first; when the fruit diminishes with the increase of the requisition, as in the inverse rule, the first and second terms must be multiplied together and divided by the third.

No proof of the rule is given, and no reference is made to the doctrine of proportion upon which it is founded. Under compound proportion is given the rule for five, seven, nine or more terms. The terms in these cases are divided into two sets, the first belonging to the argument, and the second to the requisition; the fruit in the first set is called the *produce* of the argument; that in the second is called the *divisor* of the set; they are to be transposed or reciprocally brought from one set to the other, that is, the fruit is to be put in the second set and the divisor in the first.

The Rule of Three Direct may be illustrated by the following example:

If two and a-half *palas* of saffron be obtained for three-sevenths of a *nishca*, say instantly, best of merchants, how much is got for nine *nishcas*?*

Statement:

3	5	9	
7	2	1	Answer, 52 *palas* and 2 *carshas*.

Rule of Three Inverse may be illustrated by the following examples: If a female slave, 16 years of age, bring 32 *nishcas*, what will one aged 20 cost? If an ox, which has been worked a second year, sell for 4 *nishcas*, what will one which has been worked 6 years cost?

1st question.

Statement: 16 32 20. Answer, $25\frac{3}{5}$ *nishcas*.

2d question.

Statement: 2 4 6. Answer, $1\frac{1}{3}$ *nishcas*.

In order to understand the solution it must be known that the value of living beings was supposed to be regulated by their age, the maximum value of female slaves being fixed at 16 years of age, and of oxen after 2 years' work; their relative value in the given problem being as 3 to 1. The rule of five terms may be illustrated by the following example: If the interest of a hundred for a month be five, what is the interest of sixteen for a year?

Statement:

1	12,	or transposing	1	12
100	16	the fruit,	100	16
5				5

the product of the larger set is 960, of the lesser 100; the quotient is $\frac{960}{100}$ or $\frac{48}{5}$, which is the answer.

The interest of money, judging from the examples in *Brah-*

*To understand their problems in rule of three it must be known that a *pala*=4 *carshas*; a *carsha*=16 *mashas*; and a *masha*=5 *gunjas*, or 10 *grains of barley*. Also, a *nishca*=16 *drammas*; a *dramma*=16 *panas*; a *pana*=4 *cacinis*; and a *cacini*=20 *cowry shells*.

megupta and *Lilawati*, varied from $3\frac{1}{2}$ to 5 per cent. a month, exceeding greatly the enormous interest paid in ancient Rome. It is also very high in modern India, where it is not uncommon for native merchants or tradesmen to give 30 per cent. per annum.

The rule of eleven terms may be illustrated by the following example: Two elephants which are ten in length, and nine in breadth, thirty-six in girt, seven in height, consume one *drona* of grain; how much will be the rations of ten other elephants, which are a quarter more in height and other dimensions? The fruit and denominator being transposed, the answer is $\frac{3125}{256}$. Dr. Peacock remarks that the principle of this very curious example would be rather alarming, if extended to other living beings besides elephants.

STATEMENT.

2	10
10	$\frac{50}{4}$
9	$\frac{45}{4}$
36	45
7	$\frac{35}{4}$
1	

Lucas di Borgo tells us that at his time it was usual for students in arithmetic to commit to memory one or other of two long rules which he presents. Tartaglia mentions the first of these two rules in nearly the same terms as Di Borgo, and gives also a third, which, however, differs from it only in expression. This rule formed part of the system in the practice of this subject, adapted to those who had not sufficient time to acquire, genius to comprehend, or memory to retain, the rules for the reduction and incorporation of fractions; a system reprobated by Tartaglia, and attributed by him partly to the ignorance of the ancient teachers of arithmetic at Venice, and partly to the stinginess and avarice of their pupils, who grudged the time and expense requisite for attaining a perfect understanding of the peculiarities of fractions.

An arithmetician of Verona, named Francesco Feliciano da Lazesio, objects to the memorial rules of Di Borgo as being too general in assuming that two of the quantities are of one species, and two, including the term to be found, of another species; and shows that in some cases they are all of the same denomination. He wishes to distinguish the quantities into *agents* and *patients*, and these again into *actual*, or *present*, and *future*. The first term of the proportion is the *present agent*, and its corresponding *patient* is the second; the third term is formed by the *future agent*, and its *patient* is the quantity to be determined. This, it will be noticed, is similar to the method of *cause* and *effect* adopted by some recent authors, and supposed to be original with them.

Di Borgo's method of stating and working a problem may be seen in the following example: "If a hundred pounds of fine sugar cost 24 ducats, what will be the cost of 975 pounds?"

via. v^a.

$\frac{100}{1} \times \frac{24}{1}$ ——— ——— $\frac{975}{1}$

v^a.

975
24
3900
1950
23400

0
040
03400
23400 (234 *ducati.*
10000
100
1

The following example of the same process, with fractions in every term, is given by Tartaglia: "If $3\frac{1}{2}$ pounds of rhubarb cost $2\frac{1}{3}$ ducats, what will be the cost of $23\frac{3}{4}$ pounds?"

lire. *ducati.* *lire.*

$\frac{7}{2} \times || \frac{7}{3}$ —||— $\frac{95}{4}$

3
4
12
7
Partitor 84
95
7
665
2
da partir 1330

07
49
0590
1330(15 *ducati*
844
8

000
1680(20 *grossi*
844
8

The quantities, in Di Borgo's solution, are exhibited under a fractional form, for the purpose of making the process more general, being equally applicable to fractions and whole numbers. It is sufficiently curious that he should have considered it necessary to construct the *galea* for the division by 100.

Different methods of representing the terms of the proportion were adopted by different authors. We will state a few of them as illustrating the solution of the problem, "If 2 apples cost 3 soldi, what will 13 cost?" Tartaglia states the proportion as follows:

Se pomi 2 || val soldi 3 || che valera pomi 13.

Other Italian authors write the numbers consecutively with mere spaces, and no distinctive marks between them; thus,

Pomi.	Soldi.	Pomi.
2	3	13

or thus,

1 ma.	2 da.	3 tia.
2	3	13

In Recorde and older English writers, they are written as follows:

Apples. Pence.
2 —— 3
13 ╲ 19½ Answer.

Humfrey Baker, 1562, in speaking of the rule, says, "The rule of three is the chiefest, and the most profitable, and most excellent rule of all Arithmetike. For all other rules have neede of it, and it passeth all others; for the which cause, it is sayde the philosophers did name it the Golden Rule; but now in these later days, it is called by us the Rule of Three, because it requireth three numbers in the operation." He writes the terms thus:

2 3 13

The custom which generally prevailed during the 17th century, was to separate the numbers by a horizontal line, as follows:

Apples.		Pence.		Apples.
2	——	3	——	13

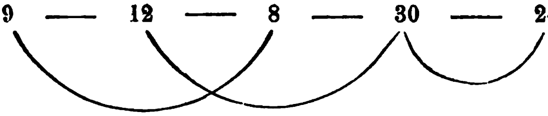
Oughtred, by whom the subject of proportion was very carefully considered, and from whom the sign, : : , to denote the equality of ratios, seems to have been derived, states a proportion as follows:

2. 3 : : 13

In still later times the simple dot which separated the terms of the ratios, was replaced by two dots, as in the form which is now universally employed.

Compound Proportion, as has been stated, was formerly included under the rule of five, six, etc., terms, there being no division of the subject into simple and compound proportion. To illustrate, take the problem, "If 9 porters drink in 8 days 12 casks of wine, how many casks will serve 24 porters 30 days?" In solving such problems Tartaglia usually puts the quantity mentioned once only in the last place but one, instead of in the second place. The statement will appear as follows:

9 —— 12 —— 8 —— 30 —— 24

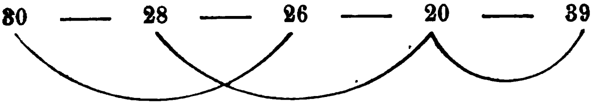


Divisor, 9×8. Dividend, $12 \times 30 \times 24$

Quotient, $\frac{8640}{72} = 120$.

The example, "Twenty braccia of Brescia are equal to 26 braccia of Mantua, and 28 of Mantua to 30 of Rimini; what number of braccia of Brescia corresponds to 39 of Rimini?" given by Tartaglia, is solved as follows:

Rimini		Mantua		Mantua		Brescia		Rimini
30	——	28	——	26	——	20	——	39



21840 780 Answer, 28.

We give another example with its solution derived from the same author. "Six eggs are worth 10 *danari*, and 12 *danari* are worth 4 thrushes, and 5 thrushes are worth 3 quails, and 8 quails are worth 4 pigeons, and 9 pigeons are worth 2 capons, and 6 capons are worth a *staro* of wheat; how many eggs are worth 4 *stara* of wheat?"

960

1—6—10—12—4—5—3—8—4—9—2—6—4

622080 Answer, 648.

ALLIGATION.—The rule for *Medial Proportion*, or *Alligation*, is of eastern origin, and appears in the *Lilawati*, though under a somewhat limited form. It is there called *suverna-ganita*, or *computation of gold*, and is applied generally to the determination of the fineness or *touch* of the mass resulting from the union of different masses of gold of different degrees of fineness. The questions mostly belong to what we call Alligation Medial. The only question given in illustration of Alligation Alternate is the following: "Two ingots of gold, of the touch of 16 and 10 respectively, being mixed together, the weight became of the fineness of 12; tell me, friend, the weight of gold in both lumps."

The rule given for the solution is, "Subtract the effected fineness from that of the gold of a higher degree of touch, and that of the one of the lower degree of touch from the effected fineness; tell me, friend, the weight of gold in both lumps? The differences multiplied by an arbitrarily assumed number, will be the weights of gold of the lower and higher degrees of purity respectively."

Statement: 16, 10. Fineness resulting, 12.

If the assumed multiplier be 1, the weights are 2 and 4 *máshas* respectively; if 2, they are 4 and 8; if $\frac{1}{2}$, they are 1 and 2: thus manifold answers are obtained by varying the assumption.

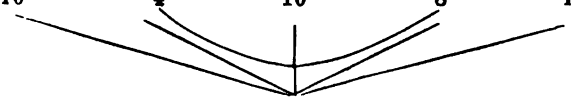
This rule, though perfectly distinct and clear, applies to two quantities only, and there is no appearance that it was ever applied to a greater number; it involves, however, the principle of the rule which is now used, recognizes the problem as unlimited, and shows in what manner an indefinite number of answers may be obtained. The extension of the rule is not entirely easy, but much more so than the invention of the original rule itself; the chief honor of the discovery of the rule belongs therefore to the mathematicians of Hindostan. The general rule was known to the Arabians and was denominated *Sekis*, a term meaning *adulterous*, inasmuch as it is not content with a single, and, as it were, *legitimate* solution of the question. It was sometimes called *Cecca* by the Italians, who appear to have known nothing further of the word than its Arabic origin; and it constitutes the *alligation alternate* of modern books of arithmetic.

The earlier Italian writers on arithmetic, in imitation of the practice of their Arabian masters, have confined the applications of this rule almost entirely to questions connected with the mixture of gold, silver, and other metals, with each other. This union was designated by the term *consolare*, which probably originated in the dreams of astrologers and alchemists, who thought it the peculiar province of the sun to produce and generate gold; and as the process of the alchemist in transmuting the baser metals into gold was supposed to be under the influence of the sun, this gradual refinement, which they in common tended to produce, was designated by the common term *consolare*. In later times, it was applied to silver as well as gold, and still more generally to the common union of these metals with copper.

To illustrate the method of Tartaglia, take the question, "A person has five kinds of wheat, worth 54, 58, 62, 70, 76 *lire* the *staro* respectively; what portion of each must be taken, so that the sum may be 100 *stara*, and the price of the mixture 66 *lire* the *staro*?"

1st. In the proportion of the numbers 10, 4, 10, 8 and 16.

54	58	62	70	76
10	4	10	8	16



66

2d. In the proportion of the numbers 14, 14, 14, 24, 24.

54,	58,	62.	70,	76,
10	10	10	12	12
4	4	4	8	8
14	14	14	4	4
			24	24

Tartaglia has given three other solutions of this example arising from a different arrangement of the ligatures. Among the English writers the method gradually assumed the form usually found in modern text-books. The method of explanation and the extension of the process as given in a few modern text-books may be ascribed to DeVolson Wood, formerly of the New York State Normal School.

POSITION.—Among the most celebrated rules to which Proportion was applied in the early text-books were those of Single and Double Position. These rules have been supplanted in this country by the simpler processes of arithmetical analysis, but they are still found in English arithmetics; and it has been suggested by a no less eminent scholar and mathematician than Dr. Hill, that they should be retained in our text-books on account of their disciplinary influences. Some historical facts concerning this old rule will be interesting to the reader.

The rule of Single Position is the only one which is found in the *Lilawati*, where it is called *Ishtacarman*, or *operation with an assumed number*. We shall give a few examples from it, which, however, present nothing very remarkable beyond the peculiarities of the mode in which they are expressed.

1. Out of a heap of pure lotus flowers, a third part, a fifth,

a sixth, were offered respectively to the gods Siva, Vishnu, and the Sun, and a quarter was presented to Bhavani; the remaining six were given to the venerable preceptor. Tell me, quickly, the whole number of flowers.

Statement: $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{4}$; known, 6.

Put 1 for the assumed number; the sum of the fractions $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{4}$, subtracted from one, leaves $\frac{1}{20}$; divide 6 by this, and the result is 120, the number required.

2. Out of a swarm of bees, one-fifth part of them settled on the blossom of the *cadamba*, and one-third on the flower of a *silind'hri;* three times the difference of these numbers flew to the bloom of a *cutaja*. One bee, which remained, hovered and flew about in the air, allured at the same moment by the pleasing fragrance of a *jasmin* and *pandanus*. Tell me, charming woman, the number of bees.

Statement: $\frac{1}{5}$, $\frac{1}{3}$, $\frac{2}{15}$: known quantity, 1; assumed 30.

A fifth part of the assumed number is 6, a third is 10, difference 4; multiplied by 3 gives 12, and the remainder is 2. Then the product of the known quantity by the assumed one, being divided by the remainder, shows the number of bees 15.

The following question is from the *Manoranjana*:

3. The third part of a necklace of pearls, broken in amorous struggle, fell to the ground; its fifth part rested on the couch, the sixth part was saved by the wench, and the tenth part was taken up by the lover; six pearls remained strung. Say of how many pearls the necklace was composed.

Statement: $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{10}$; remained, 6. Answer, 30.

Some authors have attributed the invention of the rules of position to Diophantus, though it is impossible to discover upon what grounds. When we consider the nature and difficulty of the problems solved by him, in those parts of his works which remain, we are fully justified in supposing that the Greeks had some method of analyzing and solving such problems, or they would not have proposed them in such number and variety.

The Arabs were in possession of the rules for both Single and Double Position, with all their applications, and in this instance had advanced far beyond their Indian masters; and when we consider how small were the additions which they usually made to the sciences which passed through their hands, we might very naturally be inclined to suppose that their knowledge of these rules was derived from the Greeks. There is, however, a vast gap in the history of the sciences after the time of Theon, and it is quite impossible to trace with certainty their transmission to the Arabs, or to ascertain through what channels some portion of Greek astronomy, at least, was transmitted to the Hindoos; we must therefore rest satisfied with the few hints to be gathered from authors between the 7th and 12th centuries, who had access to many writings which have since perished.

The Italian writers on arithmetic derived the knowledge of these rules directly from the Arabians, distinguishing them by the Arabic name of *El Cataym*. The questions proposed by Di Borgo and Tartaglia are of immense variety, including every case of single and double position; and the rules which are given for this purpose are such as would immediately result from the formula given in higher algebras. The following example is given and explained by Di Borgo:

4. A person buys a jewel for a certain number of *fiorini*, I know not how many, and sells it again for 50. Upon making his calculation, he finds that he gains $3\frac{1}{3}$ soldi in each *fiorino*, which contains 100 *soldi*. I ask what is the prime cost.

Suppose it to cost any sum you choose; assume 30 *fiorini*, the gain upon which will amount to 100 *soldi*, or 1 *fiorino*: 1 added to 30 makes 31; and you say that it makes 50 between capital and gain; the position is therefore false, and the truth will be obtained by saying, if 31 in capital and gain arises from a mere capital of 30, from what sum will 50 arise. Multiply 30 by 50, the product is 1500; divide it by 31, the result is $48\frac{12}{31}$; and so much I make the prime cost of the jewel.

Tartaglia says that such questions were frequently proposed as puzzles by way of dessert at entertainments, and has mixed up with his other questions a large number of such problems. The practice, from some circumstances, appears to be referable to the Greek arithmeticians of the 4th and 5th centuries, and perhaps to an earlier period.

Both Di Borgo and Tartaglia sought to include every possible case of mercantile practice under the Rule of Three, giving numerous examples and classifying them in various ways. The Italians were also the inventors of the rule of Practice, which they regarded as an application of the Rule of Three. Tartaglia gives some interesting and practical examples, with various ingenious methods of solution. The great convenience of these rules for performing the calculations which were continually occurring in trade and commerce, made them a favorite study with practical arithmeticians, and they assumed from time to time a constantly increasing neatness and distinctness of form. Stevinus, though, speaks of them with some contempt as forming "a vulgar compendium of the rule of three, sufficiently commodious in countries where they reckon by *livres*, *sous* and *deniers*." John Mellis, in his addition to Recorde's arithmetic, presents the rules of Practice in a very simple and complete form, calling attention to them as "briefe rules called rules of practise, of rare, pleasant, and commodious effect, abridged into a briefer method than hath hitherto been published." Later works gave them still greater compactness and brevity, and in Cocker's Arithmetic, published in 1677, after his death, and in others printed towards the end of the 17th century, they assumed the form in which they are now found in English arithmetics.

The subjects of Partnership and Barter, also treated by an application of Proportion, seem to have originated with the Italians. They grew out of their business transactions, and in many cases were so complicated as to require great skill and judgment in their solution. They are interesting as presenting the type of nearly all the questions of this kind found in modern text-books.

SECTION II.

THE PROGRESSIONS.

I. Arithmetical Progression.

II. Geometrical Progression.

CHAPTER I.

ARITHMETICAL PROGRESSION.

IN comparing numbers we perceive that we may have a series of numbers which vary by a common law; such a series is called a *Progression*. The more general name for such a succession of terms is *Series*, which is used to embrace every arrangement of quantities that vary by a common law, however simple or complicated, and whether expressed in numbers or in algebraic or transcendental terms. The term Progression is preferred in arithmetic, and is restricted to the arithmetical and geometrical series.

The constant relation existing between two or more successive terms of the series is called the *Law* of the progression. In the series 1, 2, 4, 8, etc., each term equals the preceding term multiplied by 2, and this constant relation constitutes the law of the series. It is evident that the law which connects the terms of a series may be greatly varied, and that we may thus have a large number of different kinds of series. The only two generally treated in arithmetic are the Arithmetical and the Geometrical series, or progressions.

Definition.—An *Arithmetical Progression* is a series of terms which vary by a constant difference; as 2, 4, 6, 8, etc. The difference between any two consecutive terms is called the *common difference.* In the series given, the common difference is 2. The common difference is sometimes called an *arithmetical ratio;* it is better, however, to restrict the use of the word ratio to a geometrical ratio, and call this what it really is, a difference.

Special attention is called to the definition of an arithmetical progression here presented. The definition usually found in our text-books is, "An arithmetical progression is a series of numbers which increase or decrease by a common difference." In the definition proposed the word *vary* is used to include both the increase and the decrease of the terms; and this is regarded as an improvement upon the old definition. It has already been adopted by two or three authors, and should be generally introduced into our text-books on arithmetic.

Notation.—The English and American authors express an arithmetical progression by writing the terms one after another with a comma between them. The French, with more precision, employ a special notation for it. They place the symbol, ÷, before the progression and the period (.) between the terms. Thus Bourdon writes,

÷2. 7. 12. 17. 22. . . 47. 52. 57. 62.

This method has been introduced into one or two American text-books, and may, in time, be generally adopted, though the tendency seems to be to adhere to the common form of expression.

Cases.—There are five quantities in an Arithmetical Progression; the *first* term, the *common difference*, the *number* of terms, the *last* term, and the *sum* of all the terms. If any three of these are given, the other two can be found from them. This gives rise to twenty different cases, in which any three terms being given, the other two may be found. These cases cannot all be solved by arithmetic, since some of them involve the solution of a quadratic equation; they are, however, very readily treated by the principles of algebra. The two principal cases in arithmetic are as follows:

1. To find the last term, having given the first term, the common difference, and the number of terms.

2. To find the sum of the terms, having given the first term, the last term, and the number of terms.

Method of Treatment.—The treatment of Arithmetical Progression in arithmetic is very simple. We derive the rule for finding the last term by noticing the law of the formation of a few terms and then generalizing this law. Thus we notice that the second term of an arithmetical progression equals the first term plus *once* the *common difference*, the third term equals the first term plus *twice* the *common difference*, etc.; hence we infer that the last term equals the first term plus the product of the *common difference* by the *number of terms less one.*

In finding the sum of the terms we take a series, then write under this series the same series in an inverted order, then adding the two series we see that twice the sum of the series is the same as the *sum of the extremes multiplied* by the *number* of terms; and generalizing this we obtain the rule for finding the sum.

In algebra we reason in the same way, except that we employ general symbols, and use a general series instead of a special one. Expressing the two fundamental rules in general formulæ, we can readily find the rest of the twenty cases by the algebraic process of reasoning. These two simple cases, I think, should in arithmetic be expressed in the concise language of algebraic symbols. Pupils who have not studied algebra will have no difficulty in understanding them. The two rules of arithmetical progression are briefly expressed thus:

$$1.\ l = a + (n - 1) \cdot d; \qquad 2.\ s = (a + l) \cdot \frac{n}{2}.$$

History.—Of the origin of the progressions and the methods of treatment, but little is known. They were the object of the particular attention of the Pythagorean and Platonic arithmeticians, who enlarged upon the most trivial properties of numbers with the most tedious minuteness. Directing their speculations, however, to the mysterious harmonics of the physical and intellectual world, they passed over, as unworthy of notice, the solutions of those problems which naturally arise from these progressions, and which appear in such numbers in Hindoo, Arabic, and modern European books on Arithmetic.

Very little is known concerning the origin of the familiar problems usually found under this subject. The problem, "How many strokes do the clocks in Venice strike in 24 hours?" is supposed to be of Venetian origin. The following familiar problem is attributed to Bede: "There is a ladder with 100 steps; on the first step is seated one pigeon, on the second step two pigeons, on the third step three, and so on increasing by one each step; tell, who can, how many pigeons were placed on the ladder." The celebrated problem,—"If a hundred stones be placed in a right line, one yard apart and the first one yard from a basket, what length of ground must a person go over who gathers them up singly, returning with them one by one to the basket?"—though found in many modern text-books, is very old, but its origin is not known.

The extraordinary magnitude of the numbers which result from the summation of a geometrical series is well calculated to excite the surprise and admiration of persons who are not fully aware of the principle upon which the increase of the terms depends; and examples are not wanting among the earliest writers, where the rash and ignorant are represented as being seduced into ruinous or impossible engagements. The most celebrated of these is that which tradition has represented as the terms of the reward demanded of an Indian prince by the inventor of the game of chess; which was a grain of wheat for the first square on the chess board, two grains for the second square, four for the third, and so on, doubling continually to sixty-four, the whole number of squares.

Lucas di Borgo solved the question, and found the result to be 18446744073709551615, which he reduces to higher denominations and finds it equal to 209022 castles of corn. He then recommends his readers to attend to this result, as they would then have a ready answer to many of those *barbioni ignari de la arithmetica* who have made wagers on such questions, and have lost their money.

CHAPTER II.

GEOMETRICAL PROGRESSION.

A GEOMETRICAL PROGRESSION is a series of terms which vary by a common multiplier; as, 1, 2, 4, 8, 16, etc. The common multiplier is called the *rate* or *ratio* of the progression; thus, in the progression given, the rate is 2. The *rate* of the progression equals the ratio of any term to the preceding term. When the progression is ascending, the rate is greater than a unit; when it is descending, the rate is less than a unit. The *rate* is by most authors called the *ratio* of the series; the reason for preferring the term *rate* will be stated presently.

Notation.—The method of writing a geometrical progression, generally employed by English and American authors, is the same as that for an arithmetical progression. The French authors, however, distinguish it from an arithmetical progression by a special notation. They place the symbol ∺ before the series, and separate the terms by a colon (:); thus,

∺ 2 : 4 : 8 : 16 : 32 : 64 : 128.

The Rate.—The constant multipler, as before stated, is generally called the *ratio* of the series. The term *rate*, it is thought, is much more appropriate and precise. The objection to the word *ratio* is that, in the comparison of numbers, the ratio is the quotient of the first term divided by the second, while the rate of a series is equal to any term divided by the previous term; hence, there is a seeming contradiction of the correct meaning of the term ratio. This contradiction may be only seeming, but to avoid all difficulty in this respect, it will be better to use a term which is appropriate and not liable to misconception. *Rate* seems to be an appropriate word, since we naturally speak of the rate of increase or decrease of anything; and by the *rate* of a progression, we mean its rate of increase or decrease.

The French mathematicians make this distinction between ratio and rate; they use the word *rapport, ratio,* in proportion, and *raison, rate,* in progression. Bourdon says, "*The constant ratio, which exists between any term and that which immediately precedes it, is called the rate of the progression.** By *rapport* they seem to mean about what we do by ratio; it is probably from the idea of *produce,* the ratio being the product of the division. Their word *raison* seems to mean the same as rate, taken probably from the idea of *cause,* the rate being the law or cause of the terms being what they are.

The term *ratio,* as used in relation to a progression, has given rise to a good deal of discussion and misapprehension. Some writers who use the word have taken the pains to tell us that they mean, not a *direct,* but an *inverse* ratio. Prof. Dodd says, when we speak of the ratio of a geometrical progression being 2, we mean that "the terms progress in a twofold ratio, which simply means that each term has the ratio of 2 to the preceding term;" and similar remarks are made by other writers. By using the word *rate* instead of *ratio,* all this difficulty and misapprehension will be avoided. It is to be hoped, therefore, that the term *rate* will be generally adopted in speaking of the law of variation of a geometrical series.

Cases.—There are five quantities considered, as in arithmetical progression; the *first* term, the *rate,* the *number* of terms, the *last* term, and the *sum* of the terms. Any three of these being given the other two can be derived from them, which gives rise to twenty distinct cases. These cannot all be solved

* Ce *rapport constant,* qui existe entre un terme et celui qui le précède immédiatement, se nomme la RAISON de la progression.—BOURDON'S Arithmetic, page 279.

by arithmetic; the first fifteen are easily derived by common algebra, and the other five readily yield to the logarithmic calculus. The two cases generally given in arithmetic are the following:

1. To find the last term, having given the first term, the rate, and the number of terms.

2. To find the sum of the terms, having given the first term, the last term, and the number of terms.

Treatment.—The general method of treatment in a geometrical progression is the same as in an arithmetical progression; and having been stated under arithmetical progression, need not be repeated here. Several cases cannot be obtained in arithmetic, since they require the solution of an equation. Four cases cannot be solved by elementary algebra, as they depend upon the solution of an exponential equation; and in obtaining the numerical results we are obliged to make use of logarithms. The two fundamental cases should, we think, in arithmetic be expressed in the symbolic language of algebra; thus,—

$$1.\ l=ar^{n-1}; \qquad 2.\ S=\frac{lr-a}{r-1}.$$

THE INFINITE SERIES.—An Infinite Series is a series in which the number of terms is infinite. In a descending progression the terms are continually growing smaller; hence if the series be continued sufficiently far, the last term must become less than any assignable quantity; and if continued to infinity, the last term must become infinitely small.

In treating an infinite series, we regard this infinitely small quantity as zero, or nothing. Thus, in finding the sum of a descending series, we use the formula $S=\frac{a-lr}{1-r}$; and regarding the last term as nothing, the term lr disappears, and we have $S=\frac{a}{1-r}$, or the sum of the terms of an infinite series descending *equals the first term divided by* 1 *minus the rate.*

This reduction of the last term to zero presents a difficulty not easily explained. The question arises, how can the last term become zero? At what point does a term become so small that, when multiplied by the rate, the product shall be nothing? To illustrate the difficulty, take the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., in which the rate is $\frac{1}{2}$. Now if this series be continued to infinity, the last term is supposed to be zero. This supposition seems to involve the idea that the term just before the last is so small that $\frac{1}{2}$ of it is nothing. Who can conceive of such a term? Who can trace the series down through all the different values, until we reach a term so small that one-half of it is nothing? This of course cannot be done. The mind shrinks from the effort; it is unable to grasp the infinitely small. Indeed, neither the infinitely great nor the infinitely small can be positively conceived; an infinite quantity and an infinitesimal are both beyond the grasp of the human mind.

What shall we do then? Shall we deny that the last term is infinitely small, or zero? Certainly not: to assume that it is not infinitely small involves a greater difficulty than the supposition that it is infinitely small. Fix upon any term, however small, and we see that it can be continually divided, and that the division will continue as long as there is a term to be divided, and can only terminate when the term becomes too small to divide, or zero. Hence, to conceive that the infinite term is not zero, is to suppose that the division stopped when it could have proceeded, which is absurd; consequently, it is absurd to suppose that the last term is not zero. The question then stands thus: we cannot comprehend that the last term is zero, and to conceive that it is not zero is absurd. We are thus in the dilemma that we must believe either the absurd or the incomprehensible. We cannot believe the absurd; we rather accept the incomprehensible. We are therefore forced to the conviction that the last term is zero, even though we cannot fully conceive it to be so. We believe that which we cannot fully understand, because not to believe it leads to an aburdity, and the mind is so constituted that it will accept the incomprehensible sooner than the absurd. We take it upon *faith;* it is the place in science "where reason falters" and faith accepts.

This method of considering the subject presents an excellent illustration of the operation of the intuitive power in many questions of religious faith. I may not be able to comprehend a first cause; but I know there must be one, or else I am involved in an absurdity, and the human mind cannot rest in the absurd. It may be remarked that the point of difficulty here considered, is one that frequently occurs in mathematics. The infinitely small is an important element in mathematical investigations. We make use of it in geometry, and in calculus it is the fundamental idea upon which the science is based.

The most satisfactory method of removing any doubt that one may have upon the assumption that the last term reduces to zero, is to take a problem which may be solved by an infinite series, and which can also be solved without it. If the result obtained by supposing the last term to be zero, agrees with the result otherwise obtained, the conclusion that the last term is zero must be accepted, whether we can conceive it or not. Such a problem is the following: "A hound and fox are 10 rods apart, and the hound pursues the fox; how far will the hound run to overtake the fox, if the latter runs $\frac{1}{10}$ as fast as the hound?"

Looking at this problem in one way, we see that when the hound has run the 10 rods the fox has run 1 rod, and they are then 1 rod apart. When the hound runs this rod, the fox has run $\frac{1}{10}$ of a rod; hence they are then $\frac{1}{10}$ of a rod apart. When the hound runs this $\frac{1}{10}$ of a rod, they are $\frac{1}{10}$ of $\frac{1}{10}$, or $\frac{1}{100}$ of a rod apart; hence the distance the hound will run to catch the fox is correctly represented by the sum of the series $10+1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+$etc., to an infinite number of terms. The sum of this series, obtained by the method of infinite series, which regards the last term as zero, equals $10\div(1-\frac{1}{10})=10\div\frac{9}{10}$

$=10\times\frac{10}{9}=11\frac{1}{9}$ rods. Hence the hound runs $11\frac{1}{9}$ rods to catch the fox.

The problem may also be solved by the following simple method of analysis: By the conditions, ten times the distance the fox runs equals the distance the hound runs; and this diminished by the distance the fox runs, is 9 times the distance the fox runs, which equals what the hound gains on the fox, or 10 rods, the distance they were apart; then once the distance the fox runs equals $\frac{10}{9}$ of a rod, and 10 times the distance the fox runs, which is the distance the hound runs, equals $10\times\frac{10}{9}=\frac{100}{9}$, or $11\frac{1}{9}$ rods. Or, we may solve it even more simply thus: the hound gains 9 rods in running 10, hence to gain 1 rod he will run $\frac{10}{9}$ of a rod, and to gain 10 rods, so as to catch the fox, he will run 10 times $\frac{10}{9}$, or $\frac{100}{9}=11\frac{1}{9}$ rods. This result corresponds with that obtained by the summation of the infinite series; hence the supposition involved in that solution, that the last term of the series equals zero, must be correct.

This problem is sometimes given as a puzzle, in which it is said that since there is always one-tenth of the previous distance between them, the hound will never catch the fox. The fallacy consists in inferring that because there is an infinite number of successive operations, it must require an infinite length of time to perform them.

A problem similar to this is the following: "A ball falls 8 feet to the floor and bounds back 4 feet, then falling bounds 2 feet, and so on; how far will it move before coming to rest?" Solving this, we find the distance to be 24 feet. It is sometimes supposed in this problem, that the body will *never* come to rest; this is a mistake, for though there will be, in theory at least, an infinite number of motions, they will be accomplished in a finite period of time. The reason of this is, that the infinitely small motions are made in infinitely small periods of time, the sum of which does not exceed a finite period.

It should be remarked that some writers maintain that the results in the infinite series are not absolutely correct, but are merely approximations; thus, that the sum of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ +etc., is not absolutely 1, but only approximately so; in other words, that all we can affirm concerning it is that it comes nearer and nearer to 1 as we increase the number of terms, though it can never reach 1. Unity is the limit towards which it is always approaching, which it never can exceed, and indeed, which it never can reach. This conception of the subject is attended with difficulties. It would seem to lead to the conclusion that in the case of the "fox and hound problem," given above, the hound would never catch the fox; unless, as a boy once remarked, "he gets near enough to grab him." So in the elastic ball dropped upon a pavement; if the result is only approximately true, does it not follow that the ball never comes to rest, but continues bounding forever? Here, as in many other cases, implicit faith in the incomprehensible is more satisfactory than a timid skepticism.

It will be interesting to notice that the two different series, $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+$etc., and $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+$etc., are each equal to the same fraction $\frac{1}{2}$. It is also an interesting truth that the sum of the series beginning with $\frac{1}{2}$, and decreasing at the rate of $\frac{1}{2}$, is just equal to 1.

SECTION III.

PERCENTAGE.

I. NATURE OF PERCENTAGE.

II. NATURE OF INTEREST.

CHAPTER I.

NATURE OF PERCENTAGE.

PERCENTAGE is a process of computation in which the basis of the comparison of numbers is a *hundred.* The same idea may also be expressed more briefly in the definition, Percentage is the process of computing in hundredths.

The former definition was first presented in one of the author's arithmetical works. Up to this time no definition had been given of Percentage as a process of arithmetic. In the text-books, the word was merely defined as meaning so many of a hundred. Soon after this publication appeared, one or two other authors adopted a definition similar to the one given above, presenting the subject as a department of the science; and in time, it is presumed, all will define it as a process of arithmetic.

It will be readily seen that Percentage has its origin in the third division of the science of arithmetic; namely, *Comparison.* We may compare numbers and determine their relations with respect to their common unit or basis. This is the first and simplest case of comparison, and gives rise to Ratio and Proportion. We may also compare numbers with respect to some number agreed upon as a basis of comparison, and develop their relations with respect to this basis. When this number is *one hundred,* we have the process of Percentage. It is thus seen that the idea of the subject presented in the definition given above is correct.

Percentage originated in the fact of the convenience of estimating by the hundred, in a decimal scale. It derives its importance and has received so full a development, partly at least, from the fact of our having a decimal currency. It occupies a more prominent place in American than in English text-books, where the money system is not decimal. Its principal use is in its application to business transactions relating to money, as will be seen in the various ways in which it is employed. It admits, however, of a purely abstract development, entirely independent of concrete examples; and is, therefore, a process of pure arithmetic.

Quantities.—Percentage embraces four distinct kinds of quantities, the *base*, the *rate*, the *percentage*, and the *amount* or *difference.*

The *Base* is the number on which the percentage is estimated. The *Rate* is the number of hundredths of the base. The *Percentage* is the result of taking a number of hundredths of the base. The *Amount* or *Difference* is the sum or the difference of the base and percentage.

The Amount and Difference are the same kind of quantities, and it would be well, in Percentage, to have some one term which would include them both. In several of the applications we have such a word; as *selling price* in Profit and Loss, *proceeds* in Discount, etc. The expression *Resulting Number* has been used, but this is a little awkward and inconvenient. The term *Proceeds*, meaning that which results or comes forth, I have sometimes thought of adopting, and indeed have adopted in one of my works. Some term, in place of amount and difference as used in percentage, is a scientific necessity, and *Proceeds* is recommended.

The *Rate* was originally expressed as a whole number, and the methods of operation based upon such expression. Latterly it is becoming the custom to represent the rate as a decimal, and to operate with it as such. This is much the better way, and will probably become universal. It gives greater simplicity to the rules, makes the treatment more scientific, and is quite as readily understood by pupils. It may be remarked that the definition of the rate will vary according to which of these forms is taken. The definition above given regards the rate as a decimal.

It will thus appear that there is a slight distinction between the term *Rate* and the expression the *rate per cent.* *Per cent.* means *by the hundred;* *rate per cent.* means a *certain number of or by the hundred;* while *Rate* means a *certain number of hundredths.* When money is loaned at 6 per cent. the *rate per cent.* is 6; but the *Rate* is .06. Thus *Rate* and *rate by the hundred,* are about identical in meaning. We may consequently define the *Rate* to be the number by which we multiply the base in order to obtain any required per cent. of it; and this is what is intended in the definition,—The rate is a number of hundredths of the base.

Cases.—It has been a question among arithmeticians under how many cases Percentage should be presented. There being four distinct classes of quantities—five, if like some authors we regard the amount and difference as distinct—any two of which being given, the others may be found, it will be seen that there are quite a large number of possible theoretical cases. What is the simplest and most scientific classification of these various cases? In other words, what are the general cases of Percentage? It has been quite customary to present the subject under *six* distinct cases, and this affords a very practical view of the subject. Authors, however, have not been uniform in their treatment. I believe that the best way is to present the subject under three general cases, each of which will contain two or three special cases, as we regard the *amount* and *difference* as one or two classes of quantities. Uniting the amount and difference under one general term, as *proceeds,* we shall have three general cases, each including two special cases, making six cases in all; regarding the *amount* and *difference* as two distinct quantities, we shall have three special cases under each general case, making nine cases in all.

These three general cases may be formally stated as follows:

1. Given, the base and the rate, to find the percentage and the proceeds.

2. Given, the base and either the percentage or the proceeds, to find the rate.

3. Given, the rate and either the percentage or the proceeds, to find the base.

Treatment.—There are two distinct methods of treatment in Percentage, which may be distinguished as the *Analytic* and the *Synthetic* methods. The *Analytic Method* consists in reducing the rate to a common fraction, and taking a fractional part of the base for the percentage, and operating similarly in the other cases. It differs particularly from the other method in the solution of the second and third cases, as will be seen by the solution of a problem. It is the method for mental analysis, and is especially suited to the subject of Mental Arithmetic. To illustrate the analytic method, take the problem, "What is 25% of 360?" We reason thus: 25% of 360 is $\frac{25}{100}$ or $\frac{1}{4}$ of 360, which is 90. To find the base take the problem,—" 90 is 25% of what number?" The solution is,—If 90 is 25%, or $\frac{1}{4}$, of some number, $\frac{4}{4}$ of the number is 4 times 90, or 360. The case of finding the rate per cent. is solved in a similar manner.

The *Synthetic Method* consists in preserving the rate in the form in which it is presented, and operating accordingly. In the synthetic method there are two ways of operating: the first consists in using the rate as a whole number, and dividing or multiplying by *a hundred;* the second operates with the rate in the form of a decimal, according to the principles of decimal multiplication and division. There has, for several years, been a tendency towards the latter method, and arithmeticians are now generally agreed in its favor.

This latter method is greatly to be preferred on account of its simplicity and scientific character. The difference may be shown by a rule for one of the cases. When the rate is used as a whole number, the rule for finding the percentage is, —*Multiply the base by the rate, and divide the product by* 100

When the rate is used as a decimal, the rule is,—*Multiply the base by the rate.* A similar difference will be found to exist in the rules for all the cases. Another consideration in favor of using the rate as a decimal is the ease with which the rules for the other cases are derived from the first. Assuming that the *percentage* equals the *base* multiplied by the *rate;* it immediately follows that the *base* equals the *percentage* divided by the *rate*, or the *rate* equals the *percentage* divided by the *base.*

To illustrate the method preferred, suppose we have the problem in Case 1.,—"What is 25% of 360?" We would reason thus: Twenty-five per cent. of 360 equals 25 hundredths times 360, or $360\times.25$, which by multiplying we find to be 90.

To illustrate Case 2, take the problem, "90 is 25% of what number?" We would solve this as follows: If 90 is 25% of some number, then some number multiplied by .25 equals 90; hence this number equals 90 divided by .25, or $90\div.25$, which by dividing we find is 360.

To illustrate Case 3, take the problem,—"90 is what per cent. of 360?" The solution is as follows: If 90 is some per cent. of 360, then 360 multiplied by *some rate* equals 90; hence the rate equals 90 divided by 360, or $90\div360$, which is .25, or 25%.

The solution of problems including the proceeds is quite similar, and need not be presented here in detail. The particular method of explanation will be found in my Higher Arithmetic.

Formulas.—These synthetic methods and rules may all be presented in general formulas, as follows:

CASE I.	CASE II.	CASE III.
1. $b\times r=p$	1. $p\div r=b$	1. $p\div b=r$
2. $b\times(1+r)=A$	2. $A\div(1+r)=b$	2. $A\div b=1+r$
3. $b\times(1-r)=D$	3. $D\div(1-r)=b$	3. $D\div b=1-r$

The 2d and 3d formulas of each case may be united in one; thus, using P for proceeds, $P=b\times(1\pm r)$; $b=P\div(1\pm r)$; $r=P\div b-1$. or $1-P\div b$.

Applications.—The applications of Percentage are very extensive, owing to the great convenience of reckoning by the hundred in financial transactions. These applications are of two general classes; those not including the element of time, and those which include this element. The following are the most important of these two classes of applications:

1st Class.	2d Class.
1. Profit and Loss.	1. Simple Interest.
2. Stocks and Dividends.	2. Partial Payments.
3. Premium and Discount.	3. Discounting.
4. Commission.	4. Banking.
5. Brokerage.	5. Exchange.
6. Insurance	6. Equation of Payments.
7. Taxes.	7. Settlement of Accounts.
8. Duties and Customs.	8. Compound Interest
9. Stock Investments.	9. Annuities.

The different cases of the first class are solved as in pure percentage, and the rules are almost identical, the technical terms being substituted for *base, percentage*, etc. The solutions of the various cases of the second class are somewhat modified by the introduction of the element of time. The development of these various cases would occupy too much space for this work, and moreover does not constitute a part of the philosophy of arithmetic; we shall, therefore, give only a single chapter on the general nature of Interest.

CHAPTER II.

NATURE OF INTEREST.

PERCENTAGE embraces two general classes of problems, —those that involve the element of time, and those that do not involve this element. The most important application of percentage into which this element enters is Interest; and indeed all such applications may be embraced under this general term.

Interest may be defined as money paid, or charged for the use of money. It is usually reckoned as so many units on a hundred, and is thus included under the general process of Percentage. The sum upon which interest is reckoned is called the *Principal*, in distinction from the interest or profit, which is subordinate to it. The sum of the interest and principal is called the *Amount*.

Interest is either *Simple* or *Compound*. Simple Interest is that which is reckoned or allowed upon the principal only, during the whole time of the loan. Compound Interest is reckoned, not only on the sum loaned, but also on the interest as it becomes due. Interest unpaid is regarded as a new loan upon which interest should be paid.

Simple Interest.—In considering the subject of simple interest, the primary object is to find the interest on a given principal for a given time and rate. Various methods have been devised for the solution of this problem. The simplest in principle and most natural, is to find the interest for one year by multiplying the principal by the rate, and multiplying this interest by the time expressed in years. The objection to this method in practice arises from the fact that the time is often given in months and days, which frequently reduce to an inconvenient fractional part of a year. This difficulty has led to a modification of the rule proposed above, which is known as the method of "aliquot parts."

The importance of a method that can be readily applied in business, has led to the exercise of considerable ingenuity in order to discover the shortest and simplest rule in practice. The method now regarded as the simplest is that known as the "six per cent." method. It is based on the rate of 6%, which is the usual rate in this country, and may be expressed as follows: *Call half the number of months cents, and one-sixth of the number of days mills, and multiply their sum, which will be the interest of $1 for the rate and time, by the principal.* Another way of stating this rule is,—*Regard the months as cents, and one-third of the days as mills, and multiply their sum by one-half of the principal.* For short periods a modification of the rule, which may be popularly expressed,—*Multiply dollars by days and divide by* 6000, is the most convenient in practice, and is very generally employed by business men. There are also many other methods of working interest which need not be stated here.

The general method of finding the interest of a principal may be expressed in a general formula as in Percentage. The general formula is $i=ptr$, which is readily remembered by the sentence which it suggests—"I equals Peter." The several cases which arise in interest can be readily derived from this fundamental formula. These several rules may be expressed as follows:

1. $i=ptr$. 3. $t=i\div pr$.
2. $p=i\div tr$. 4. $r=i\div pt$.

It is objected to the "six per cent. method," that it gives too great an interest, since it reckons only 360 days in a year; and it has been suggested that to compute the interest on a loan by this method would be to take usury, and in some states would result in a forfeiture of the debt, or some other penalty. This seems like putting a very nice point on the matter, though it is true that the six per cent. method gives a little more interest than when we reckon 365 days to the year. To obtain exact interest, we find the interest for the years, multiply the interest of one year by the number of days, and divide by 365, and take the sum of the two results. A full presentation of the applications of interest to business and the latest methods of treatment may be found in the author's Higher Arithmetic.

Rates of Interest.—It is a noteworthy fact that the propriety of receiving interest for the use of money, has been questioned. Indeed, the practice has been censured in both ancient and modern times as an immorality and a wrong to society. It may seem that so absurd a notion hardly needs a passing notice, for it is clear that a similar objection may be made to the charge of rents, or even to profits of any kind. A capitalist may invest his money in business and receive a certain return for it; and if he chooses to let some one else invest it and have the care of such investment, it is clear that he should receive some remuneration for surrendering to another the profit he might have made himself. Again, the borrower can with capital secure a large return of profit in business, and is not only entirely willing to pay for the use of such capital, but is in equity under obligations to do so. Interest on loans is, therefore, a benefit to both the borrower and lender; and should therefore be both required and allowed.

The rate of interest is determined strictly by the principle of competition. When the capital to be invested exceeds the demands of borrowers, the rate of interest is low; when the demand is in excess of the capital, the rate will be high. The rate will vary also with the security of the loan; thus the rate on landed mortgages is usually lower than on property less secure and certain, and consequently state loans are usually made at low rates. A lender assumes that he must be paid something for the risk of a loan, and that the greater the risk the greater the charge. It is on this principle that high interest is often said to be synonymous with bad security. A high rate of interest may also be due to large profits on capital. In a community where the returns on capital are large, as in rich mining districts for instance, all who have capital would desire to invest, and consequently the difficulty of obtaining a loan would increase and higher rates would obtain. In such cases the opportunity for large gains by the capitalist and the increased demand by the borrower would both conspire to increase the rate of interest.

The rates of interest have usually been regulated by governments. This action is founded upon a variety of reasons. It has been argued that lenders are unproductive consumers of part of the profit which is produced by labor. Such a notion leaves out of sight, however, that production is impossible without capital, and that capital is accumulated and employed with a view to profit. It is also held that if the state does not regulate rates, borrowers will be open to fraud and extortion on the part of unprincipled lenders. This is the principal consideration in favor of state control of interest rates; and yet there are valid if not unanswerable objections to it. It is, of course, the duty of the government to protect the citizen against usury and fraud; but most of the considerations in favor of regulating rates of interest will apply to the regulation of the prices of food, land, wages, etc. It seems to be a growing opinion that capital should seek investment at rates determined by natural laws of demand and supply, as the prices of other property are regulated, and not be controlled by legislative enactment.

Historical.—The payment of interest on money has been the custom from very early times. We learn from the New Testament that it was paid on bankers' deposits in Judea, though the Jews were forbidden by the laws of Moses to exact interest from one another. In Europe, interest was alternately prohibited and allowed, the church being generally hostile to the practice. In Italy, the trade in money was recognized, and the custom of borrowing and lending was common. In England, it was first sanctioned by the Parliament in 1546, the rate being fixed at 10 per cent.; but in 1552 it was again prohibited. Mary, however, borrowed at 12 per cent., which appears to have been the usual rate at that period at Antwerp. In 1571, it was again made legal at 10 per cent., a rate at which the Scotch Parliament fixed it in 1587. The rate fell at the beginning of the seventeenth century, James I. having borrowed in Denmark at 6 per cent. In 1624, it was reduced to 8 per cent.; in 1651, to 6 per cent.; in 1724, to 5 per cent., at which legal rate it remained until all usury laws were repealed, an event which occurred only a few years ago. In 1773, it was limited to 12 per cent. in India. In 1660, the rate in Scotland and Ireland was from 10 to 12 per cent.; in France 7 per cent.; in Italy and Holland 3 per cent.; in Spain from 10 to 12 per cent.; in Turkey 20 per cent.; but the East India Company, while the legal rate was 6 per cent., continued to borrow at 4 per cent.

The term *Usury*, meaning the "use of a thing," was originally applied to the legitimate profit arising from the use of money, and meant merely the taking of interest for money. Laws were established in various countries fixing the amount of interest or usury, and the evasion of these laws by charging excessive usury, led to the present use of the term. By the old Roman law of the Twelve Tables, the rate of interest allowed as legitimate was the *usura centesima,* which was strictly 1 per cent. a month; and has been supposed by some to have amounted to 12, and by others to 10 per cent. a year. The Roman laws against excessive usury were frequently renewed and constantly evaded, and the same is true of other countries. In England, during the reign of Henry VIII., 10 per cent. was allowed; by 21 James I., 8 per cent.; by 12 Charles II., 8 per cent.; by 12 Anne, 5 per cent. Subsequently to the passage of the latter act, the usury laws were relaxed by several statutes, and they were ultimately repealed in 1854. Any rate of interest, however high, may now be legally stipulated for, but 5 per cent. remains the legal interest recoverable on all contracts, unless otherwise specified.

Much concern has been shown by governments in attempting to fix rates of interest, and prevent usury. The legislation of Solon relieved the Athenian mortgagors; and during many years of the Roman Republic, the regulation of loans, the limitation of the rate of interest, and the relief of insolvent debtors, formed a perpetual topic of agitation, and finally of legislation. In most of the European countries the administration has busied itself, from time to time, in fixing rates of interest, and in denouncing or forbidding usurious bargains. Such legislation has, however, proved vain; for while the most stringent laws were in force, high rates of interest on loans were common, the law being incompetent to provide against evasion of the statute.

The legal rate of the United States government is 6 per cent. Each State fixes its own rate, and attaches its special penalties for usury. In several of the States the usury laws have been repealed, and the general tendency is to allow an open market to the investment of capital.

Origin of Methods.—The importance of a knowledge of the principles of interest, discount, etc., led arithmeticians to notice these subjects at an early day. Interest was early divided into *Simple* and *Compound.* Compound Interest was properly called *usura*, and was rarely practised in the transactions of merchants with each other. Stevinus terms compound interest, *interest prouffitable*, or *celuy qu'on ajouste au capital*, whilst the corresponding discount is termed *interest dommageable*, or *celuy qu'on soubstrait du capital.*

Problems in simple interest were by Tartaglia and his predecessors, solved by the Rule of Three. In calculating the interest of a sum from one day to another, the determination of the number of days in the interval seemed somewhat embarrassing, and Tartaglia gives a rule for this purpose of which he seems somewhat proud. In passing from one city of Italy to another an additional source of embarrassment presented itself in the different days on which the year was supposed to commence, being reckoned at Venice from the 1st of March, at Florence from the Annunciation of the Virgin, and in most other cities of Italy from Christmas day.

Tartaglia has noticed five methods of finding the amount of a sum of money at compound interest. Suppose the question to be to find the amount of L300 for 4 years at 10 per cent. *a capo d'anno;* the first method is by the following four statements:

$$100 : 300 : : 110 : 330$$
$$100 : 330 : : 110 : 363$$
$$100 : 363 : : 110 : 399\tfrac{3}{10}$$
$$100 : 399\tfrac{3}{10} : : 110 : 439\tfrac{23}{100}.$$

The second method merely replaces 100 and 110 by 10 and 11 in the proportion; the third, which is his own method, multiplies 300 four times successively by 11, and divides the last product by 10,000; the fourth consists in adding four successive tenths to the principal; the last in calculating the amount for L100, and then finding the amount of L300, or any other proposed sum, by a simple proportion.

With the exception of discount at compound interest and its application to correct in part the conclusion respecting the values of annuities, there are few, if any, other questions of compound interest which Tartaglia and his contemporaries can be said to have resolved. A very natural difficulty arose in the solution of questions of this kind: "What is the interest of 100 for 6 months, interest being reckoned at the rate of 20 per cent. per annum?" Lucas di Borgo and others made out that this would be 10; that is, they calculated that, simple interest only being allowed, it was a matter of indifference into how many portions of time the whole period was divided, whether into months or half-years.

Lucas di Borgo has an article on calculating tables of interest in which he speaks of their great utility, thereby showing that such tables were in use in Italy, although no work of that date containing them is known to be extant. The first compound interest tables now known are those which are presented by Stevinus in his arithmetic, which give the present worth of 10,000,000 from 1 to 30 years, in sixteen tables, the interest being reckoned successively from 1 to 16 per cent., and in eight other tables, where the interest is differently reckoned, according to the custom of Flanders.

The origin of the various modern methods of calculating interest is not known. The method by "aliquot parts" is a favorite rule of the English arithmeticians, and probably originated with them. The "six per cent. method" has been attributed to a Mr. Adams, author of a work on arithmetic. The particular form of the six per cent. method popularly stated, "multiply dollars by days and divide by 6000," was used among business men before it was introduced into any arithmetic, and is presumed to have had its origin in some counting-house, but it is not known where.

SECTION IV.

THE THEORY OF NUMBERS.

I. Nature of the Subject.

II. Even and Odd Numbers.

III. Prime and Composite Numbers.

IV. Perfect, Imperfect, etc., Numbers.

V. Divisibility of Numbers.

VI. Divisibility by the Number Seven.

VII. Properties of the Number Nine.

CHAPTER I.

NATURE OF THE SUBJECT.

THE Theory of Numbers, as generally presented, embraces the classification and investigation of the properties of numbers. This subject has engaged the attention and enlisted the talents of many celebrated mathematicians. The ancient writers, who did little for the development of arithmetic as a science or an art, spent much time in theorizing upon the properties of numbers. The science of arithmetic with them was mainly speculative, abounding in fanciful analogies and mysterious properties.

Pythagoras attributed to numbers certain mystical properties, and seems to have conceived the idea of what are now termed Magic Squares. Aristotle, amongst other numerical speculations, noticed the practice, in almost all nations, of dividing numbers into groups of tens, and attempted to give a philosophical explanation of the cause. The earliest regular system of numbers is that given by Euclid in the 7th, 8th, 9th, and 10th books of his "Elements," which, notwithstanding the embarrassing notation of the Greeks, and the inadequacy of geometry to the investigation of numerical properties, is still very interesting, and displays, like all other parts of the same celebrated work, that depth of thought and accuracy of demonstration for which its author is so eminently distinguished.

Archimedes, also, paid particular attention to the powers and properties of numbers. His tract, entitled "Arenarius," contains a method of multiplying and dividing which bears a considerable analogy to that which we now employ in multiplication and division of powers, and which some modern writers have thought inculcated the principles of our present system of logarithms. Before the invention of algebra, however, but little progress could be made in this branch of the science; accordingly we find that comparatively few principles had been discovered until the time of Diophantus. This eminent mathematician, who is the author of the most ancient existing work on the subject of algebra, presents many interesting problems in the properties of numbers; but, owing to the difficulties of a complicated notation and a deficient analysis, little progress was made, compared with the advance of modern times.

From the time of Diophantus the subject remained unnoticed, or at least unimproved, until Bachet, a French analyst, undertook the translation of Diophantus into Latin. This work, which was published in 1621, contained many marginal notes of the translator, and may be considered as presenting the first germs of our present theory. These were afterward considerably extended by Fermat, in his posthumous edition of the same work, published in 1670, which contains many of the most elegant theorems in this branch of analysis; but they are generally left without demonstration, which he explains in a note by saying that he was preparing a treatise of his own upon the subject. Legendre accounts for the omission by saying that it was in accordance with the spirit of the times for learned men to propose problems to each other for solution. They generally concealed their own method in order to obtain new triumphs for themselves and their nation; and there was about this time an especial rivalry between the English and French mathematicians. Thus it has happened that most of the demonstrations of Fermat have been lost, and the few that remain only make us regret the more those that are wanting.

The most of these theorems remained undemonstrated until the subject was again renewed by Euler and Lagrange. Euler, in his "Elements of Algebra," and some other publications, demonstrated many of the theorems of Fermat, and also added some interesting ones of his own. Lagrange, in his additions to Euler's Algebra and in other writings, greatly extended the theory of numbers by the discovery of many new properties. The subject has received its largest contributions, however, from the hands of Gauss and Legendre.

Legendre, in his great work, "Essai sur la Théorie des Nombres," was the first to reduce this branch of analysis to a regular system. Gauss, in his "Disquisitiones Arithmeticæ," opened a new field of inquiry by the application of the properties of numbers to the solution of binomial equations of the form, $x^n - 1 = 0$, on the solution of which depends the division of the circle into n equal parts. This solution he accomplished in several partial cases; whence the division of the circle into a prime number of equal parts is performed by the solution of equations of inferior degrees; and when the prime number is of the form $2^n + 1$ the same may be done geometrically—a problem that was far from being supposed possible before the publication of the work mentioned.

The most celebrated English work on the subject is that of Peter Barlow, published in 1811, from the preface of which most of the preceding historical facts have been culled. It presents a clear and concise statement of the principles of the subject, and contains several original contributions, among which may be mentioned a demonstration of Fermat's general theorem on the impossibility of the indeterminate equation $x^n \pm y^n = z^n$, for every value of n greater than 2. This demonstration, however, has been tacitly ignored by mathematicians; and the French Institute and other learned societies have continued to propose the problem for solution.

Almost every modern mathematician of eminence, however, has contributed more or less to the advancement of the theory. In the collected works of Euler, Gauss, Jacobi, Cauchy, Dirichlet, Lagrange, Eisenstein, Poinsot, and others, numerous memoirs on the subject will be found; whilst the recent mathematical journals and academical transactions contain researches in the same field, by all the ablest living mathematicians. One of the most complete treatises on the subject is that of Prof. H. J. S. Smith in the article entitled, "Reports on the Theory of Numbers," which commenced in the *Transactions of the British Association* for 1859. It embraces a lucid, critical history of the subject, rendered doubly valuable by copious references to the original sources of information.

It will be seen from this brief statement that the subject of the theory of numbers is one of great magnitude and difficulty, requiring the application of the principles of algebra for its development. It is, therefore, not appropriate to treat of it in this work, except so far as to show its logical relation to the general divisions of the science, and to present a few simple properties that may be readily understood by means of the ordinary principles of arithmetic. These will be interesting to young arithmeticians, and perhaps the means of cultivating a taste for a more thorough study of the subject.

The subjects to which the attention of the reader will be briefly directed are the following:

1. Even and Odd Numbers.
2. Prime and Composite Numbers.
3. Perfect, Imperfect, etc., Numbers.
4. Divisibility of Numbers.
5. Divisibility by the Number Seven.
6. Properties of the Number Nine.

CHAPTER II.

EVEN AND ODD NUMBERS.

NUMBERS have been divided into many different classes, founded upon peculiarities discovered by investigating their properties. The series 1, 2, 3, 4, etc., is called the series of *Natural Numbers*. The Natural Numbers are classified with respect to their relation to the number *two*, into *Odd* and *Even* numbers. They are also divided into two classes with respect to their composition, called *Prime* and *Composite* numbers. Composite Numbers are divided into two classes, *Perfect* and *Imperfect* numbers, this classification being based upon the relation of the numbers to the sum of their factors. Imperfect Numbers are also divided into two classes with respect to the numbers being greater or less than the sum of their factors. Numbers which are equal each to the sum of the divisors of the other, are called *Amicable Numbers*. A few remarks will be made on each one of these classes.

Of the various classes of numbers, the simplest and most natural division is that of *Even* and *Odd* numbers. This division is founded upon the relation of numbers to the number 2. *Even* numbers are those which are multiples of 2; *Odd* numbers are those which are not multiples of 2. In the series of natural numbers the increase is by a unit; in the series of even numbers the scale of increase is dual. The former arise from counting by 1's, beginning with the unit; the latter in counting by 2's, beginning with the duad. The even numbers are divided into the *oddly even* numbers, 2, 6, 10, 14, etc.; and the *evenly even* numbers, 4, 8, 12, 16, etc. The odd numbers are divided into the *evenly odd* numbers 1, 5, 9, 13, etc; and the *oddly odd* numbers, 3, 7, 11, 15, etc.

The formula for the even numbers is $2n$; the formula for the odd numbers is $2n+1$. In the oddly even numbers n is an odd number; in the evenly even numbers n is an even number. In the evenly odd numbers n is even; in the oddly odd numbers n is odd. The evenly odd numbers are of the form $4n+1$; the oddly odd numbers are of the form $4n+3$.

There are many interesting principles relating to even and odd numbers, a few of which will be stated.

1. Every prime number except 2 is an odd number.

2. The differences of the successive square numbers produce the odd numbers.

3. The sum or difference of two even numbers or two odd numbers is an even number.

4. The sum or difference of an even number and an odd number is odd.

5. The sum of any number of even numbers is even; the sum of an even number of odd numbers is even, and the sum of an odd number of odd numbers is odd.

6. The product of two even numbers is even; of two odd numbers is odd; of an even number and an odd number is even.

7. The quotient of an even by an odd number, when exact, is even; the quotient of an odd by an odd, when exact, is odd; the quotient of an even by an even, when exact, is either even or odd.

8. An odd number is not exactly divisible by an even number, and the remainder is odd.

9. If an even number is not exactly divisible by an even number, its remainder is even.

10. If an even number is not exactly divisible by an odd number, then when the quotient is even the remainder is even, and when the quotient is odd, the remainder is odd.

11. If an odd number is not exactly divisible by an odd number, then when the quotient is odd the remainder is even, and when the quotient is even the remainder is odd.

12. If an odd number divides an even number, it will also

divide one-half of it; if an even number be divisible by an odd number, it will be divisible by double that number.

13. Any power of an even number is even; and conversely the root of an even number which is a complete power is even.

14. Any power of an odd number is odd; and conversely the root of an odd number which is a complete power is odd.

15. The sum or difference of any complete power and its root is even.

These principles can be readily proved by the ordinary methods of arithmetical reasoning. To illustrate, take the third principle, the reasoning of which is as follows: Two even numbers are each a *number of* 2's, hence their sum will be the sum of two different *numbers of* 2's, which must be a *number of* 2's, and their difference will be the difference between two different *numbers of* 2's, which is also a *number of* 2's. In adding two odd numbers we will have a *number of* 2's+1, added to another *number of* 2's+1, which will give us a *number of* 2's+2, or an *exact number* of 2's, etc.

The simplest method is by using the general notation of algebra. Thus in the given principle, these two even numbers will be represented by $2n$ and $2n'$; their sum will be $2n+2n'$, or $2(n+n')$, which is of the form of $2n$, and is thus even; their difference will be $2n-2n'$, or $2(n-n')$, which is of the form of $2n$, and is even. The two odd numbers are of the form $2n+1$ and $2n'+1$, and their sum is $2(n+n'+1)$, which is of the form of $2n$, and even; their difference is $2n-2n'$, or $2(n-n')$, which is evidently even. All the other principles may be demonstrated in a similar manner.

CHAPTER III.

PRIME AND COMPOSITE NUMBERS.

THE most celebrated classification of numbers is that of *Prime* and *Composite.* This classification is with respect of their formation by multiplication or the possibility of their being resolved into factors. The Composite number is one which can be produced by the multiplication of other numbers; the Prime number is one which cannot be produced by the multiplication of other numbers. The distinction may be regarded as having reference to the dependence or independence of their existence. The composite number is regarded as deriving its existence from other numbers which make it; the prime number does not derive its being from any other numbers, but is independent and self-existent.

Perhaps no subject in arithmetic has received more attention from mathematicians than that of Prime and Composite Numbers. The object has been to discover some general method of finding prime numbers, and of determining whether a given number is prime or composite. Such a method, though laboriously sought for by the best mathematical minds, has not, beyond a certain limit, been discovered.

The problem of ascertaining prime numbers was discussed as far back as the days of Eratosthenes, a mathematician of Alexandria, distinguished also as having first conceived the plan of measuring the earth. He invented a method of obtaining primes by excluding from the series of natural numbers those that are not prime, and thus discovering those that are. This method consisted in inscribing the series of odd numbers upon parchment, and then cutting out the composite numbers, and leaving the primes. The parchment, with its holes, resembled a *sieve;* hence the method is called *Eratosthenes' sieve.* His method may be illustrated as follows:

Suppose we write the series of odd numbers from 1 to 99 inclusive. Since the series increases by 2, the third term from 3 is $3+3\times2$, which is divisible by 3; hence every third term is divisible by 3, and is therefore composite. In a similar manner we see that every fifth term after 5 is divisible by 5, and therefore composite; and every seventh term after 7 is divisible by 7, and therefore composite. Cutting out these composite numbers, we have all the prime numbers below 100. By this method, assisted by some mechanical contrivance, Vega computed and published a table of prime numbers from 1 to 400,000.

This method is, however, very tedious and inconvenient, and mathematicians have earnestly sought for properties of prime and composite numbers to guide them in ascertaining primes. The following principles are useful in discovering or determining prime numbers:

1. All prime numbers except 2 are *odd,* and consequently terminate with an odd digit. The converse of this, that all odd numbers are prime, is not, however, true.

2. All prime numbers, except 2 and 5, must terminate with 1, 3, 7, or 9; all other numbers are composite. This is the series of odd digits with the omission of 5, since any number terminating with 5, can be divided by 5 without a remainder.

3. Every prime number, except 2, if increased or diminished by 1, is divisible by 4. In other words, every prime number, except 2, is of the form $4n\pm1$. This will admit of demonstration.

4. Every prime number, except 2 and 3, if increased or diminished by 1, is divisible by 6. In other words, every prime number, except 2 and 3, is of the form $6n\pm1$. This may also be demonstrated.

5. Every prime number, except 2, 3, and 5, is a measure of

the number expressed, in common notation, by as many 1's as there are units, less one, in the prime number. Thus, 7 is a measure of 111,111; and 13 of 111,111,111,111.

6. Every prime number, except 2 and 5, is contained without a remainder in the number expressed in the common notation by as many 9's as there are units, less one, in the prime number itself. Thus, 3 is a measure of 99; 7 of 999,999; and 13 of 999,999,999,999.

7. Three prime numbers cannot be in arithmetical progression, unless their common difference is divisible by 6; except 3 be the first prime number, in which case there may be three prime numbers in such progression, but in no case can there be more than three.

8. This last principle is generally true, and may be stated as follows: There cannot be n prime numbers in arithmetical progression unless their common difference be divisible by $2 . 3 . 5 . 7 . 11 . . . n$; except the case in which n is the first term of the progression, in which case there may be n such numbers, but not more.

Though we have no general method for finding prime numbers, there are several ways of detecting whether an assigned number is or is not a prime. Several remarkable formulas have been discovered which contain a large number of prime numbers. The formula x^2+x+41, by making successively $x=0$, 1, 2, 3, 4, etc., will give a series 41, 43, 47, 53, 61, 71, etc., the first forty terms of which are prime numbers. This formula is mentioned by Euler in the Memoirs of Berlin, 1772. Of the two formulas x^2+x+17, and $2x^2+29$, the former gives seventeen of its first terms primes, and the latter twenty-nine. Fermat asserted that the formula 2^m+1 is always a prime when m is taken any term in the series 1, 2, 4, 8, 16, etc.; but Euler found that $2^{32}+1=641\times 6,700,417$ is not a prime.

One of the most celebrated theorems for investigating primes is that discovered by Fermat and known as *Fermat's Theorem.* The theorem may be stated thus: If p *be a prime, the* (p—1)*th* *power of every number prime to* p *will, when diminished by unity, be exactly divisible by* p. Expressed in algebraic language, we have the theorem $P^{p-1}-1$, is a multiple of p when p and P are prime to each other. Thus, 25^6-1 is exactly divisible by 7.

Fermat is said to have been in possession of a proof of the theorem, though Euler was the first to publish its demonstration. Euler's first demonstration was a very simple one, and is that usually given in the text-books. Amongst the other demonstrations of the theorem, those given by Lagrange are highly esteemed.

It has been demonstrated by Legendre (*Essai sur la Théorie des Nombres*), that every arithmetical progression, of which the first term and common difference are prime to each other, contains an infinite number of prime numbers. It has been also shown by him that if N represents any number, then will the formula

$$\frac{N}{h.log\,N-1.08366}$$

represent the number of prime numbers that are less than N, *very nearly.*

Another celebrated theorem is that invented by Sir John Wilson, known as *Wilson's Theorem.* This theorem may be stated as follows: *The continued product, increased by unity, of all the integers less than a given prime, is exactly divisible by that prime.* The algebraic formula which expresses the theorem, $1+1.2.3\ldots(n-1)$, is divisible by n, n being a prime number. Thus $1+1.2.3.4.5.6=721$, is exactly divisible by 7.

This theorem was first demonstrated by Lagrange; his process of reasoning, as might be expected, was very ingenious. It was afterward demonstrated by Euler, and finally by Gauss, who extended the theorem by proving that "*The product of all those numbers less than, and prime to, a given number,* $a\pm1$, *is divisible by a;*" the ambiguous sign being —, when a

is of the form p^m, or $2p^m$, p being any prime number greater than 2; and, also, when $a=4$; but positive in all other cases.

Wilson's Theorem furnishes us with an infallible rule, in theory, for ascertaining whether a given number be a prime or not; for it evidently belongs exclusively to those numbers, as it fails in all other cases; but it is of no use in a practical point of view, on account of the great magnitude of the product even for a few terms.

In the later works on the Theory of Numbers it is demonstrated that, *No algebraical formula can represent prime numbers only.* It is also shown that, *The number of prime numbers is infinite.* The latter proposition is evident *a priori;* the former was pretty nearly evident from induction before it received a rigid demonstration.

The distribution of prime numbers does not follow any known law; but for a given interval it is found that the number of primes is generally less the higher the beginning of the interval is taken. The whole number of primes below 10,000 is 1,230; between 10,000 and 20,000 it is 1,033; between 20,000 and 30,000 it is 983; between 90,000 and 100,000 it is 879. The largest prime which had been verified when Barlow wrote, is $2^{31}-1=2,147,483,647$, which was found by Euler.

The term *prime* is also applied to a species of numbers called *complex numbers*, first suggested by Gauss in 1825. According to this theory, a *complex integer* is of the form $a+b\sqrt{-1}$, in which a and b denote ordinary (real) integers. The product a^2+b^2, of a complex number $a+b\sqrt{-1}$, and its *conjugate*, $a-b\sqrt{-1}$, is called its *norm*, and is denoted by the symbols $N(a+b\sqrt{-1})$, $N(a-b\sqrt{-1})$. The four *associative* numbers, $a+b\sqrt{-1}$, $a\sqrt{-1}-b$, $-a-b\sqrt{-1}$, and $-a\sqrt{-1}+b$, as well as their respective conjugates, have all the same norm. A complex number is said to be *prime* when it admits of no divisor except itself, its associatives, and the four units, 1, -1, $\sqrt{-1}$, and $-\sqrt{-1}$. Many of the higher theorems, such as that of Fermat, may be extended to the system of complex numbers.

CHAPTER IV.

PERFECT, IMPERFECT, ETC., NUMBERS.

HAVING separated numbers into their factors, the human mind, ever active in the attempt to discover the new, began to compare the sum of the factors or divisors of numbers with the numbers themselves, and thus discovered certain relations which gave rise to three new classes of numbers. In some cases it was seen that a number was just equal to the sum of all of its divisors, not including itself, and such numbers were called Perfect Numbers. Numbers not possessing this property were called Imperfect Numbers; and were divided into two classes, Defective and Abundant, according as they were greater or less than the sum of their divisors.

Pushing the comparison still further, it was also discovered that some numbers were reciprocally equal to their divisors; and this relation was so intimate that such numbers were regarded as friendly or Amicable Numbers. These several classes will be formally defined in this chapter. Perfect Numbers were discovered by Michael Steiffel, Professor of Mathematics in the University of Jena; Amicable Numbers were first investigated by the Dutch mathematician Van Schooten, who lived from 1581 to 1646.

A *Perfect Number* is one which is equal to the sum of all its divisors, except itself; thus, $6=1+2+3$; $28=1+2+4+7+14$ An *Imperfect Number* is one which is not equal to the sum of all its divisors. Imperfect Numbers are *Abundant* or *Defective.* An *Abundant Number* is one the sum of whose divisors exceeds the number itself; as, $1+2+3+6+9>18$. A

Defective Number is one the sum of whose divisors is less than the number itself; as, $1+2+4+8<16$.

Every number of the form (2^{n-1}) (2^n-1), the latter factor being a prime number, is a perfect number. The only values of n yet found, which make 2^n-1 a prime are 2, 3, 5, 7, 13, 17, 19, and 31; there are, therefore, only ten perfect numbers known. Substituting 2 for n in the formula, we have $2(2^2-1)=6$, the first perfect number; the second is $2^2(2^3-1)=28$. The first eight perfect numbers are, 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128. An author gives also 2417851639228158837784576, 9903520314282971830-448816128. Each number, as is seen, ends in 6 or 28.

The difficulty in finding perfect numbers consists in finding primes of the form of 2^n-1. The greatest prime number, according to Barlow, yet ascertained, is $2^{31}-1=2147483647$, discovered by Euler; and the last of the above perfect numbers, which depends upon this, is the greatest perfect number known at present; and Barlow remarks that it is probably the greatest that will ever be discovered; for, as they are merely curious without being useful, it is not likely that any person will attempt to find one beyond it. An author of an arithmetic gives two other numbers which are said to be perfect, but I have not tested them and do not know his authority.

Two numbers are called *Amicable* when each is equal to the sum of the divisors of the other; thus, 284 and 220. The formulas for finding amicable numbers are $A=2^{n+1}d$ and $B=2^{n+1}bc$, in which n is an integer, and b, c, and d are prime numbers satisfying the following conditions: 1st, $b=3\times 2^n-1$; 2d, $c=6\times 2^n-1$; 3d, $d=18\times 2^{2n}-1$. If we make $n=1$, we find $b=5$, $c=11$, and $d=71$; substituting these in the above formulas, we have $A=4\times 71=284$, and $B=4\times 5\times 11=220$, the first pair of amicable numbers. The next two pairs are 17296, 18416, and 936358, 9437056.

The first pair, 220 and 284, were found by E. Van Schooten, with whom the name *amicable* appears to have originated, though

Rudolphus and Descartes were previously acquainted with this property of certain numbers. A formula for amicable numbers was, in fact, given by Descartes, and afterwards generalized by Euler and others.

Figurate Numbers.—Figurate Numbers are numbers formed from an arithmetical progression whose first term is unity, and common difference integral, by taking successively the sum of the first two, the first three, the first four, etc., terms of the series; and then operating on the new series in the same manner as in the original progression in order to obtain a second series, and so on.

For example, take the series of natural numbers in which the common difference is 1, as represented by A in the margin; then the series B, derived as stated above, will be figurate numbers; series C, derived as above from series B and series D, derived from series C, will be figurate numbers. Other series could be obtained by beginning with any other arithmetical series whose first term is 1, and common difference an integer. Thus, the series derived from the progression 1, 3, 5, 7, 9, etc., is 1, 4, 9, 16, 25, etc.

A, 1-2-3-4-5-6-7
B, 1-3-6-10-15-21-28
C, 1-4-10-20-35-56-84
D, 1-5-15-35-70-126-210

A more general method of conceiving figurate numbers is to regard them as a series of numbers, the general term of each series being expressed by the formula,

$$\frac{n(n+1)(n+2)(n+3) \quad . \quad . \quad . \quad . \quad (n+m)}{1 \, . \, 2 \, . \, 3 \, . \, 4 \quad . \quad . \quad . \quad (m+1)},$$

in which m represents the order of the series, and n represents the place of the required term.

Series of figurate numbers are divided into *orders;* when $m = 0$, the series is of the 1st order; when $m = 1$, the series is of the 2d order; when $m = 2$, it is of the 3d order, etc.

By regarding m equal to 0 in this formula, and substituting successive numbers 1, 2, 3, etc., for n, it will be seen that the general term is n, and we find that the figurate series of the first order is the series of natural numbers, 1, 2, 3, 4, etc., n.

By regarding m equal to 1, the general term of the series becomes $\frac{n(n+1)}{1 \cdot 2}$, and substituting the successive values of n, 1, 2, 3, etc., we find the terms to be 1, 3, 6, 10, 15, 21, 28, etc., which is the series of figurate numbers of the second order.

In a similar manner we find the general term of the figurate series of the 3d and 4th orders to be respectively,

$$\frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} \text{ and } \frac{n(n+1)(n+2)\,(n+3)}{1 \cdot 2 \cdot 3 \cdot 4};$$

from which we can readily derive those series. These several series of figurate numbers are the same as those represented in the margin above.

One of the most remarkable properties of the series of figurate numbers is that, if the nth term of a series of any order be added to the $(n+1)$th term of the series of the preceding order, the sum will be equal to the $(n+1)$th term of the series of the given order. Thus, in the series marked C, if we add the second term, 4, to the third term, 6, in series B, we shall have the third term, 10, of series C; the third term of series C plus the fourth term of series B equals the fourth term of series C, etc.

If we begin with a series of 1's, all of the series of figurate numbers may be deduced in succession by the application of this principle.

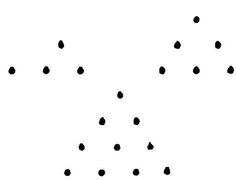
ORDERS OF FIGURATE NUMBERS.

Series of 1's	1,	1,	1,	1,	1,	1,	1,	1,	1,	1
1st order	1,	2,	3,	4,	5,	6,	7,	8,	9,	10
2d order	1,	3,	6,	10,	15,	21,	28,	36,	45,	55
3d order	1,	4,	10,	20,	35,	56,	84,	120,	165,	220
4th order	1,	5,	15,	35,	70,	126,	210,	330,	495,	715
5th order	1,	6,	21,	56,	126,	252,	462,	792,	1287,	2002
6th order	1,	7,	28,	84,	210,	462,	924,	1716,	3003,	5005
7th order	1,	8,	36,	120,	330,	792,	1716,	3432,	6435,	11440

By inspecting these series, it will be seen that the values ead diagonally upward are the numerical coefficients of the terms in the development of $(a+b)$ with an exponent corresponding to the order of the series. It is said that it was this principle which gave rise to a complete investigation of the subject of figurate numbers.

In speaking of defining figurate numbers by giving the form of each of the orders, Barlow remarks that it is more simple to deduce the generation of figurate numbers from their form than to deduce their form from their generation. The principle given above, showing the relation of the terms of two successive orders of figurate numbers, is ascribed to Fermat, and is considered by him as one of his most interesting propositions.

Polygonal Numbers are figurate numbers which represent the sides of polygons. The second series of figurate numbers, 1, 3, 6, 10, etc., are called *triangular numbers*, because the number of units that they express can be arranged in the form of a triangle. If we take the series 1, 3, 5, 7, 9, etc., in which the common difference is 2, we obtain



the figurate series, 1, 4, 9, 16, 25, etc., which are called *square numbers*, because they can be arranged in a square. The series 1, 4, 7, 10, etc., in which the common difference is 3, gives the series 1, 5, 12, 22, etc., which are called *pentagonal numbers*, because they can be arranged in the form of a pentagon. In a similar manner we obtain *hexagonal*, *heptagonal*, *octagonal*, etc., numbers. It will be noticed that the number of the sides of the polygon which they represent is always two greater than the common difference of the series from which they were derived. When the common difference of the series in arithmetical progression is 1, the sums of the terms give the *triangular* numbers; when

Common difference=1;	1,	2,	3,	4,	5,	6
Triangular numbers	1,	3,	6,	10,	15,	21
Common difference=2;	1,	3,	5,	7,	9,	11
Square numbers	1,	4,	9,	16,	25,	36
Common difference=3;	1,	4,	7,	10,	13,	16
Pentagonal numbers	1,	5,	12,	22,	35,	51

the common difference is 2, the sums of the terms are the *square* numbers; when the difference is 3, the sums are the *pentagonal* numbers, and so on.

These numbers are called *polygonal* from possessing the property that the same number of points may be arranged in the form of that polygonal figure to which it belongs. Thus the pentagonal numbers 5, 12, 22, 35, 51, etc., may be severally arranged in the form of a pentagon. Thus, 5 points will form one pentagon; 12 points will form a second pentagon enclosing the former; 22, a third pentagon enclosing both of the former, etc.

The following property of polygonal numbers was discovered by Fermat: *Every number is either a triangular number or the sum of two or three triangular numbers; every number is either a square number, or the sum of two, three, or four square numbers; every number is either a pentagonal number or the sum of two, three, four, or five pentagonal numbers;* etc. This property is generally true, although it has been demonstrated for only triangular and square numbers. All the other cases still remain without demonstration, notwithstanding the researches of many of the ablest mathematicians. Fermat himself, however, as appears from one of his notes on Diophantus, was in possession of the demonstration, although it was never published, which circumstance renders the theorem still more interesting to mathematicians, and the demonstration of it more desirable.

Pyramidal Numbers are those which represent the number of bodies that can be arranged in pyramids. They are formed by the successive sums of polygonal numbers in the same manner as the polygonal numbers are formed from arithmetical progressions. The *Triangular Pyramidal* numbers are the series of figurate numbers derived from the series of triangular numbers. Thus, from the triangular numbers 1, 3, 6, 10, 15, etc., we have the triangular pyramidal numbers 1, 4, 10, 20, etc. The *Square Pyramidal* numbers are derived from the square numbers.

CHAPTER V.

DIVISIBILITY OF NUMBERS.

IN factoring a composite number, we divide successively by exact divisors of the number till we obtain a quotient which is a prime number. In order to know by what numbers to divide, it is convenient to have some tests of divisibility, otherwise it would be necessary to try several numbers until we hit upon one which is exactly contained. There are certain laws which indicate, without the test of actual division, whether a number is divisible by a given factor, some of which are simple and may be readily applied. The investigation of these laws of the relations of the factors of numbers to the numbers themselves, gives rise to a subject known as the *Divisibility of Numbers.*

The laws for the divisibility of numbers, as usually presented, embrace the conditions of divisibility by the numbers 2, 3, 4, etc., up to 12. These laws may be stated as follows:

1. *A number is divisible by* 2 *when the right-hand term is zero or an even digit.* For, the number is evidently an even number, and all even numbers are divisible by 2.

2. *A number is divisible by* 3 *when the sum of the numbers denoted by its digits is divisible by* 3. It will be shown hereafter that every number is a multiple of 9, plus the sum of its digits; hence, since 3 is a factor of 9, the number is divisible by 3 when the sum of the digits is divisible by 3.

3. *A number is divisible by* 4, *when the two right-hand terms are ciphers, or when they express a number which is divisible by* 4. If the two right-hand terms are ciphers, the number

equals a number of hundreds, and since 100 is divisible by 4, any number of hundreds is divisible by 4. If the number expressed by the two right-hand digits is divisible by 4, the number will consist of a number of *hundreds*, plus the number expressed by the two right-hand digits; and since both of these are divisible by 4, their sum, which is the number itself, is divisible by 4.

4. *A number is divisible by* 5, *when its right-hand term is* 0 *or* 5. If the right-hand term is 0, the number is a number of times 10; and since 10 is divisible by 5, the number itself is divisible by 5. If the right-hand term is 5, the entire number will consist of a number of *tens*, plus 5; and since both of these are divisible by 5, their sum, which is the number itself, is divisible by 5.

5. *A number is divisible by* 6, *when it is even and the sum of the digits is divisible by* 3. Since the number is even, it is divisible by 2, and since the sum of the digits is divisible by 3, the number is divisible by 3, and since it contains both 2 and 3 it will contain their product, 3×2, or 6.

6. *A number is divisible by* 7, *when the sum of the odd numerical periods, minus the sum of the even numerical periods, is divisible by* 7. The law for the divisibility by 7 is perhaps of not so much practical importance as the others, being not quite so readily applied, but it is of too much scientific interest to be omitted from the series. Its demonstration will be given in the following chapter.

7. *A number is divisible by* 8, *when the three right-hand terms are ciphers, or when the number expressed by them is divisible by* 8. If the three right-hand terms are ciphers, the number equals a number of *thousands;* and since 1000 is divisible by 8, any number of thousands is divisible by 8. If the number expressed by the three right-hand digits is divisible by 8, the entire number will consist of a number of *thousands*, plus the number expressed by the three right-hand digits (thus 17368 =17,000+368); and since both of these parts are divisible by 8, their sum, which is the number itself, is divisible by 8.

8. *A number is divisible by* 9, *when the sum of the digits is divisible by* 9. This law is derived from showing that a number may be resolved into two parts, one part being a multiple of 9 and the other the sum of the digits. A complete demonstration is presented on a subsequent page, to which the reader is referred.

9. *A number is divisible by* 10, *when the unit term is* 0. For, such a number equals a number of *tens*, and any number of tens is divisible by 10; hence the number is divisible by 10.

10. *A number is divisible by* 11, *when the difference between the sums of the digits in the odd places and in the even places is divisible by* 11, *or when the difference is* 0. This law is derived by showing that a number may be resolved into two parts, one part being a multiple of 11, and the other part consisting of the sum of the digits in the odd places, minus the sum of the digits in the even places. A complete demonstration will be presented on a subsequent page.

11. *A number is divisible by* 12, *when the sum of the digits is divisible by* 3 *and the number expressed by the two right-hand digits is divisible by* 4. For, since the sum of the digits is divisible by 3, the number is divisible by 3, and since the number expressed by the two right-hand digits is divisible by 4, the number is divisible by 4; hence, since the number is divisible by both 3 and 4, it is divisible by their product, or 12.

These laws are simple, and, with the exception of those relating to the numbers 7, 9, and 11, readily applied. The laws of dividing by 9 and 11 present some interesting points, which will be formally discussed. It will be noticed, upon examining text-books on arithmetic, and also works on the theory of numbers, that the law of divisibility by 7 is omitted. Apparently efforts were made to discover such a law, for several writers give some special rules for dividing by 7; but it would seem that no general law was known to them. In the principle as above presented, this hiatus is filled up by a law not quite so simple as that for the other numbers, but still of scientific interest, if not of much practical value. Besides the law given, there are several other laws, interesting as showing the development of the subject, and which we therefore present. The methods of demonstration are similar to those used in proving the divisibility of numbers by 9 and 11; indeed, one of the laws from which the others were derived was discovered by the application of that method to the number 7. I shall therefore first present the demonstration of divisibility by 9 and 11, and then state and demonstrate the laws relating to the number 7.

Divisibility by Nine.—The law of divisibility by nine has been known for a long time. By whom it was discovered has not been ascertained. Its application to testing the correctness of the work in the fundamental rules, called proof by "casting out nines," has been attributed to the Arabs. The law, as previously stated, is that a number is divisible by nine when the sum of the digits is divisible by nine. This principle depends on a more general law which will be first stated, and then the law of exact division, as well as some other interesting principles, will be drawn from it.

1. *A number divided by* 9 *leaves the same remainder as the sum of the digits divided by* 9.

This theorem can be demonstrated both arithmetically and algebraically. We will first present the arithmetical demonstration. If we take any number, as 6854, and analyze it, as in the margin, we will see that it consists of two parts; the first part a *multiple* of 9, and the second part the *sum of the digits.*

$$6854 = \begin{cases} 4 = 4 \\ 50 = 5\times 10 = 5\times(9+1) = 5\times 9 + 5 \\ 800 = 8\times 100 = 8\times(99+1) = 8\times 99 + 8 \\ 6000 = 6\times 1000 = 6\times(999+1) = 6\times 999 + 6 \end{cases}$$

$$\therefore 6854 = \underbrace{5\times 9 + 8\times 99 + 6\times 999}_{\text{Multiple of 9}} + \underbrace{4+5+8+6}_{\text{Sum of digits}}$$

The first part is evidently divisible by 9, hence the only remainder that can arise from dividing a number by 9 will be equal to the remainder arising from dividing the sum of the digits by 9. When the sum of the digits is exactly divisible by 9, it is evident that the number itself is exactly divisible by 9, which proves the theorem. From this theorem the following principles may be readily inferred:

2. *A number is exactly divisible by* 9 *when the sum of its digits is divisible by* 9.

3. *The difference between any number and the sum of its digits is divisible by* 9.

4. *A number divided by* 9 *gives the same remainder as any one formed by changing the order of the figures.*

5. *The difference between two numbers, the sums of whose digits are equal, is exactly divisible by* 9.

The fundamental theorem may also be demonstrated algebraically as follows: Let a, b, c, d, etc., represent the digits of any number, and r the *radix* of the scale, that is, the number of units in a group; then every number may be represented by formula (1) below. If we now subtract b, c, d, etc., from one part of this expression, and add them to another part, it will not change the value, and we shall have formula (2); and factoring, we obtain formula (3).

(1). $N=a+br+cr^2+dr^3+er^4+$etc.

(2). $N=br-b+cr^2-c+dr^3-d+er^4-e$, etc. $+a+b+c+d+e$ +etc.

(3). $N=b(r-1)+c(r^2-1)+d(r^3-1)+e(r^4-1)$ +etc. $+a$ $+b+c+d+e+$etc.

Now, $r-1$, r^2-1, r^3-1, etc., etc., are all divisible by $r-1$; hence the only remainder which can arise from dividing the number by $r-1$, will occur from dividing $a+b+c+d+$etc., by $r-1$; that is, any number divided by $r-1$ leaves the same remainder as the sum of the digits divided by $r-1$. In our decimal scale $r=10$, hence $r-1=9$; and hence any number divided by 9 leaves the same remainder as the sum of the digits divided by 9. This law is the basis of some very interesting properties, and also of the proof of the fundamental rules called "casting out nines."

Divisibility by Eleven.—The law of the divisibility of numbers by 11 is quite similar to that of 9. This might have been anticipated, as they each differ from the basis of the scale by unity, the former being a unit below and the latter a unit above the base. The law, as previously stated, is that a number is divisible by 11 when the difference between the sum of the digits in the odd places and the even places is divisible by 11. This principle depends upon a more general one, which will first be stated, and then this, as well as some other interesting principles, will be derived from it.

1. *Every number is a multiple of* 11, *plus the sum of the digits in the odd places, minus the sum of the digits in the even places.* This principle may be demonstrated both arithmetically and algebraically. We will first give the arithmetical proof. If we take any number, as 65478, and analyze it as in-

$$65478=\begin{cases} 8= & & & +8 \\ 70= & 7\times10= & 7\times(11-1)= & 7\times11-7 \\ 400= & 4\times100= & 4\times(99+1)= & 4\times99+4 \\ 5000= & 5\times1000= & 5\times(1001-1)= & 5\times1001-5 \\ 60000= & 6\times10000= & 6\times(9999+1)= & 6\times9999+6 \end{cases}$$

$$\therefore\ 65478=\overbrace{7\times11+4\times99+5\times1001+6\times9999}^{\text{Multiples of 11.}}+\overbrace{8+4+6}^{\text{Sum of odd digits.}}-\overbrace{5+7}^{\text{Sum of even digits.}}$$

dicated, we shall see that it consists of two parts; the first being a multiple of 11, and the second consisting of the sum of the digits in the odd places, minus the sum of the digits in the even places. The first part is evidently divisible by 11; hence the only remainder that can arise from dividing a number by 11 will be equal to the remainder arising from dividing the difference between the sums of the digits in the odd places and the even places by 11. When this difference is exactly divisible by 11, it follows that the number itself is divisible by 11. When the sum of the digits in the even places is greater than the sum in the odd places, we take the difference, divide by 11, and subtract the remainder from 11 to find the true remainder. The reason for this will appear from the above demonstration. From this theorem the following principles can be readily inferred:

2. *A number is exactly divisible by* 11, *when the sum of the*

digits in the odd places is equal to the sum of the digits in the even places.

3 *A number is exactly divisible by* 11, *when the difference between the sums of the digits in the odd places and the even places is a multiple of* 11.

4. *A number increased by the sum of the digits in the even places and diminished by the sum of the digits in the odd places, is exactly divisible by* 11.

5. *The excess of* 11's *in any number is not changed by adding any multiple of* 11 *to the sum of the digits of either order.*

The algebraic demonstration of this property is as follows: Taking the same formula as for the number 9, we add b and then subtract b, we subtract c and then add c, etc., the formula becoming (2) below, being the same in value as the first, but changed in form. Then, factoring, we have (3).

(1). $N=a+br+cr^2+dr^3+er^4+\text{etc.}$

(2). $N=br+b+cr^2-c+dr^3+d+er^4-e+\text{etc.}+a-b+c-d+e,$ etc.

(3). $N=b(r+1)+c(r^2-1)+d(r^3+1)+e(r^4-1)+\text{etc.}+(a+c+e+\text{etc.})-(b+d+\text{etc.})$

Now $r+1$, r^2-1, r^3+1, etc., are each divisible by $r+1$; hence the only remainder that can arise from dividing this number by $r+1$ must arise from dividing $(a+c+e+\text{etc.})-(b+d+\text{etc.})$ by $r+1$; that is, by dividing the *difference of the sum of the digits in the even places subtracted from the sum of the digits in the odd places* by $r+1$. In the decimal scale, $r=10$, and $r+1=11$; hence we see that any number divided by 11 leaves the same remainder as the difference of the sum of the digits in the even places, subtracted from the sum of the digits in the odd places does when divided by 11. When this difference is exactly divisible by 11, the number itself is divisible, which proves the principle of the divisibility by 11. This principle may also be used for the proof of the fundamental rules, but not quite so conveniently as that of the number 9.

CHAPTER VI.

THE DIVISIBILITY BY SEVEN.

THE *Divisibility of Numbers*, as presented by different authors, embraces the conditions of divisibility by the numbers 2, 3, etc., up to 12, with the omission of the number 7. This omission leads us to inquire whether there is any general law for the divisibility of numbers by 7. A few of our text-books present some special truths in regard to this subject, among which are the following:

1. *A number is divisible by 7 when the unit term is one-half or one-ninth of the part on the left.* Thus 21, 42, 63, 126, and 91, 182, 273, etc.

2. *A number is divisible by 7 when the number expressed by the two right-hand terms is five times the part on the left, or one-third of it.* Thus 525, 840, 1995, and 602, 903, 3612, etc.

3. *A number consisting of not more than two numerical periods is divisible by 7 when these periods are alike.* Thus 45045, 235235, 506506, etc., are divisible by 7.

There are, however, some general laws for the divisibility by 7, which seem to have been overlooked by most writers on the theory of numbers, and which, though of not much practical importance, are interesting in a scientific point of view. The first and least simple of these laws is as follows:

1. *A number is divisible by 7, when the sum of once the first, or units digit, 3 times the second, 2 times the third, 6 times the fourth, 4 times the fifth, 5 times the sixth, once the seventh, 3 times the eighth, etc., is divisible by 7.* It will be

seen that the series of multipliers is 1, 3, 2, 6, 4, 5. To illustrate the law, take the number 7935942, and we have for the sum of the multiples of the digits, $1\times 2+3\times 4+2\times 9+6\times 5+4\times 3+5\times 9+1\times 7=126$, which is exactly divisible by 7; and if we divide the number itself by 7, we find there is no remainder. Assuming this principle—it will be demonstrated on page 398—we can derive several other principles of divisibility from it.

In this law we see that the second half of the series of multipliers, 6, 4, 5, equals respectively 7 minus the first half, 1, 3, 2; hence, instead of adding the multiples of the second series, 6, 4, 5, we may subtract the respective multiples of the terms of the second period by the first series of multipliers, 1, 3, 2, which will give rise to the following principle:

2. *A number is divisible by 7, when the number arising from the sum of once the first digit, 3 times the second, 2 times the third, minus the sum of the same multiples of the next three digits, plus the sum of the same multiples of the next three digits, etc., is divisible by 7.*

It will be seen that the series of multipliers is 1, 3, 2, the first products additive, the second products subtractive, etc.; the odd numerical periods being additive and the even periods subtractive. If we take the number 5439728, we have $1\times 8+3\times 2+2\times 7-1\times 9-3\times 3-2\times 4+1\times 5=7$, which is divisible by 7. Upon trial we find the original number is also exactly divisible by 7.

This second principle may also be stated thus: *A number is divisible by 7 when the sum of the multiples expressed by the numbers, 1, 3, 2, of the terms of the odd numerical periods, minus the sum of the same multiples of the terms of the even numerical periods, is divisible by 7.*

Now, if we add exact multiples of 7 to the multiples of the terms which are united in the test of divisibility, it will not change the remainder. Thus, taking the number 5439728, if we add 7×2 to 3×2, we have 10×2, or 20; and adding 98×7

to 2×7 we have 100×7, or 700; hence we may use in place of $1\times8+3\times2+2\times7$, $8+20+700$, or 728, the first numerical period; and in the same way it may be shown that we may use the second period subtractively in the test, etc. Hence from Principle 2 we may derive the following principle:

3. *A number is divisible by 7, when the sum of the odd numerical periods, minus the sum of the even numerical periods, is divisible by 7.*

To illustrate, take the number 5,643,378,762; we have for the sum of the odd numerical periods $762+643=1405$; for the sum of the even periods, $378+5=383$; the difference is 1022, which is exactly divisible by 7; and if we divide the number itself by 7, we find that there is also no remainder.

If we apply the same reasoning to Principle 1, by which we derived Principle 3 from Principle 2, we shall derive from it the following principle:

4. *A number is divisible by 7, when the sum of the numbers denoted by the double numerical periods is divisible by 7.* Thus, in the number 5,643,378,762, we have $5,643+378,762=384,405$, which is divisible by 7, and the number is also divisible by 7.

The first principle, from which I have derived the other three, may be demonstrated arithmetically and algebraically.

Let us take any number as 98765432 and analyze it thus:

$$\begin{array}{rrrr}
2= & & & 1\times2 \\
30= & 3\times10= & 3\times(7+3)= & 3\times7+3\times3 \\
400= & 4\times100= & 4\times(98+2)= & 4\times98+2\times4 \\
5000= & 5\times1000= & 5\times(994+6)= & 5\times994+6\times5 \\
60000= & 6\times10000= & 6\times(9996+4)= & 6\times9996+4\times6 \\
700000= & 7\times100000= & 7\times(99995+5)= & 7\times99995+5\times7 \\
8000000= & 8\times1000000= & 8\times(999999+1)= & 8\times999999+1\times8 \\
90000000= & 9\times10000000= & 9\times(9999997+3)= & 9\times9999997+3\times9
\end{array}$$

Here $98765432=$ a multiple of 7 plus *once* the 1st term, plus *three* times the second term, plus *two* times the third term, plus *six* times the fourth term, plus *four* times the fifth term, plus *five* times the sixth term, plus *once* the seventh term, plus *three* times the eighth term. Hence the only remainder that can occur must arise from dividing the sum of the multiples of the terms by 7; hence when the sum of these multiples is divisible by 7, the number is divisible by 7, which proves the principle.

The second principle, which is readily derived from the first, may be demonstrated independently, as follows:

$$
\begin{array}{rrrr}
2= & & & 1\times2 \\
30= & 3\times10= & 3\times(7+3)= & 3\times7+3\times3 \\
400= & 4\times100= & 4\times(98+2)= & 4\times98+2\times4 \\
5000= & 5\times1000= & 5\times(1001-1)= & 5\times1001-1\times5 \\
60000= & 6\times10000= & 6\times(10003-3)= & 6\times10003-3\times6 \\
700000= & 7\times100000= & 7\times(100002-2)= & 7\times100002-2\times7 \\
8000000= & 8\times1000000= & 8\times(999999+1)= & 8\times999999+1\times8 \\
90000000= & 9\times10000000= & 9\times(9999997+3)= & 9\times9999997+3\times9
\end{array}
$$

Here 98765432=a multiple of 7, plus *once* the first digit, plus *three* times the second, plus *twice* the third, minus *once* the fourth, minus *three* times the fifth, minus *twice* the sixth, plus *once* the seventh, plus *three* times the eighth. Hence the only remainder that can occur must arise from dividing the difference between the additive and subtractive multiples of the digits by 7; therefore, when this difference is divisible by 7, the number is divisible by 7, which proves the principle. When the sum of the subtractive multiples of the digits is greater than the sum of the additive, we take the difference, divide by 7, and subtract the remainder from 7 to find the true remainder.

To demonstrate the third principle, take any number, as 7,946,321,675 and analyze it, and it will be seen to consist of parts which are multiples of 7, plus the periods in the odd places, minus the periods in the even places.

$$
7946321675=\left\{
\begin{array}{rrr}
675= & & 675 \\
321000= & 321\times(1001-1)= & 321\times1001-321 \\
946000000= & 946\times(999999+1)= & 946\times999\times1001+946 \\
7000000000= & 7\times(1000000001-1)= & 7\times999001\times1001-\ \ 7
\end{array}
\right.
$$

$$
\underbrace{321\times1001+946\times999999+7\times1000000001}_{\text{Multiples of 7.}}+\underbrace{675+946}_{\text{Odd periods.}}-\underbrace{321+7}_{\text{Even periods.}}
$$

Now 1001 is a multiple of 7, 999999 is 999 times 1001, and 1000000001 is also a multiple of 1001, and if we continue the number to still higher periods, we shall find a constant series of multiples of 1001, alternately 1 more and 1 less than the number represented by one unit of the period. Hence 7,946,321,675 is composed of the sum of three multiples of 7, plus (675+946) — (321+7), or the difference between the sums of the even and odd periods. The first part is evidently divisible by 7, therefore the divisibility of the number depends on the divisibility of the difference of the sums of the odd and even periods; and when this difference is divisible by 7, the number itself must be divisible by 7, which proves the principle.

From this demonstration, we can immediately derive the following principle, more general than the one stated and from which that may be derived:

5. *Any number divided by 7 gives the same remainder as is obtained when the sum of the odd numerical periods, minus the sum of the even numerical periods, is divided by 7.* If the sum of the even periods is the greater, we find the difference, divide by 7, and subtract the remainder from 7 for the true remainder.

This investigation leads to a still more general principle of divisibility, derived from the fact that 1001, which may be considered as the basis of the above demonstration, is the product of 7, 11, and 13; hence what we have just proved for 7, is also true of 11 and 13. The most general form of the principle then is as follows:

6. *Any number divided by* 7, 11, *or* 13 *gives the same remainder as is obtained when the sum of the odd numerical periods, minus the sum of the even numerical periods, is divided by* 7, 11, *or* 13 *respectively.*

A special truth growing out of this general principle, had been previously given in the rule that any number of not more than two periods, when those two periods are alike, is divisible by 7, 11, or 13. All such numbers, on examination, will be found to be multiples of 1001, and, of course, divisible by its factors. It may seem surprising that those who were familiar with this special truth, and were thus on the very brink of a discovery, did not extend it and reach the general law above presented.

The fourth Principle, which was derived from the first, may also be demonstrated independently by a method similar to that used in proving the third Principle. The algebraic demonstration of Principle 1, which is the foundation of the other principles, is as follows: Take the same general formula as used in demonstrating the divisibility by 9 and 11, add and subtract $3b$, 3^2c, 3^3d, etc., and the formula is readily reduced to the form of (5).

(1). $N=a+br+cr^2+dr^3+er^4+fr^5+gr^6+hr^7+\text{etc.}$

(2). $N=br-3b+cr^2-3^2c+dr^3-3^3d+er^4-3^4e+fr^5-3^5f$, etc. $+a+3b+9c+27d+81e+243f+\text{etc.}$

(3). $N=b(r-3)+c(r^2-3^2)+d(r^3-3^3)+e(r^4-3^4)+f(r^5-3^5)$ $+g(r^6-3^6)$ etc. $+a+3b+9c+27d+81e+243f+729g$, etc.

(4). $N=b(r-3)+c(r^2-3^2)+d(r^3-3^3)+e(r^4-3^4)+f(r^5-3^5)$ $+g(r^6-3^6)+\text{etc.}+a+3b+\left\{\begin{matrix}7c\\2c\end{matrix}\right.+\left\{\begin{matrix}21d\\6d\end{matrix}\right.+\left\{\begin{matrix}77e\\4e\end{matrix}\right.+\left\{\begin{matrix}238f\\5f\end{matrix}\right.+$ $\left\{\begin{matrix}728g\\g\end{matrix}\right.+\text{etc.}$

(5). $N=\{b(r-3)+c(r^2-3^2)+d(r^3-3^3)+e(r^4-3^4)+f$ $(r^5-3^5)+g(r^6-3^6)+\text{etc.}+7c+21d+77e+238f+728g+\text{etc.}\}+a$ $+3b+2c+6d+4e+5f+1g+\text{etc.}$

Now the first part of this expression is exactly divisible by $r-3$, or 7; hence the only remainder that can arise must occur from dividing $a+3b+2c+6d$, etc., by $r-3$, or 7; that is, by dividing by 7 the sum of *once* the first digit, *three* times the second, *two* times the third, *six* times the fourth, *four* times the fifth, *five* times the sixth, and so on in the same order; and when this sum is exactly divisible by 7, the number is divisible by 7. By a slight change in the terms of the formula, the theorem as stated in the second form may also be derived.

Several years after the discovery of the law expressed in Principle 2, I learned that Prof. Elliott had employed the same property as early as 1846. Whether it was known to any mathematicians previous to this date, I am not able to ascertain.

Laws for Other Numbers.—In a similar manner we may find a law for the divisibility of numbers by 13, 17, etc. The law for 13 may be stated as follows: *A number is divisible by* 13 *when* ONCE *the first term,* MINUS *the sum of* 3 *times the second* 4 *times the third and* 1 *time the fourth,* PLUS *the sum of the same multiples of the next three terms,* MINUS *the sum of the same multiples of the next three terms, etc., is divisible by* 13.

It will be noticed that after the first term, the series of numbers by which we multiply is 3, 4, 1, which is easily remembered and readily applied. To illustrate, take the number 8765432; we have $2-(3\times3+4\times4+1\times5)+(3\times6+4\times7+1\times8)=26$, which is divisible by 13; and on trial we find the number itself is also divisible.

This law is derived from the more general principle that any number divided by 13 will give the same remainder as that obtained by dividing the result arising from the above multiples by 13. This principle may be demonstrated by taking any number, as 4987654, and analyzing it as in the previous case.

$$4987654=\begin{cases} 4= & & & +1\times4\\ 50= & 5\times10= & 5\times(13-3)= & 5\times13-3\times5\\ 600= & 6\times100= & 6\times(104-4)= & 6\times104-4\times6\\ 7000= & 7\times1000= & 7\times(1001-1)= & 7\times1001-1\times7\\ 80000= & 8\times10000= & 8\times(9997+3)= & 8\times9997+3\times8\\ 900000= & 9\times100000= & 9\times(99996+4)= & 9\times99996+4\times9\\ 4000000= & 4\times1000000= & 4\times(999999+1)= & 4\times999999+1\times4 \end{cases}$$

Laws for the divisibility of numbers by 17, 19, 23, etc., may be obtained in a similar manner. We present a few of them below, including 7, 11, and 13, already given.

$$7\ldots\begin{cases} 1,\ 3,\ 2,\ -1,\ -3,\ -2,\ 1,\ 3,\ 2,\ -1,\ -3,\ -2, \text{ etc.}\\ \text{or } 1,\ 3,\ 2,\ 6,\ 4,\ 5,\ 1,\ 3,\ 2,\ 6,\ 4,\ 5, \text{ etc.} \end{cases}$$

$$11\ldots\begin{cases} 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1, \text{ etc.}\\ \text{or } 1,\ 10,\ 1,\ 10,\ 1,\ 10,\ 1,\ 10,\ 1,\ 10,\ 1,\ 10, \text{ etc.} \end{cases}$$

$$13\ldots\begin{cases} 1,\ -3,\ -4,\ -1,\ 3,\ 4,\ 1,\ -3,\ -4,\ -1,\ 3,\ 4, \text{ etc.}\\ \text{or } 1,\ 10,\ 9,\ 12,\ 3,\ 4,\ 1,\ 10,\ 9,\ 12,\ 3,\ 4, \text{ etc.} \end{cases}$$

$$17\ldots\begin{cases} 1,\ -7,\ -2,\ -3,\ 4,\ 6,\ -8,\ 5,\ -1,\ 7,\ 2,\ 3, \text{ etc.}\\ \text{or } 1,\ 10,\ 15,\ 14,\ 4,\ 6,\ 9,\ 5,\ 16,\ 7,\ 2,\ 3, \text{ etc} \end{cases}$$

$$41\ldots\begin{cases} 1,\ 10,\ 18,\ 16,\ -4,\ 1,\ 10,\ 18,\ 16,\ -4, \text{ etc.}\\ \text{or } 1,\ 10,\ 18,\ 16,\ 37,\ 1,\ 10,\ 18,\ 16,\ 37, \text{ etc.} \end{cases}$$

73. . .	1, 10, 27, −22, −1, −10, −27, 22, 1, 10, 27, etc.
	or 1, 10, 27, 51, 72, 63, 46, 22, 1, 10, 27, etc.
99. . .	1, 10, 1, 10, 1, 10, 1, 10, 1, 10, 1, 10, etc.
101. . .	1, 10, −1, −10, 1, 10, −1, −10, 1, 10, −1, −10, etc.
	or 1, 10, 100, 91, 1, 10, 100, 91, 1, 10, 100, 91, etc.

The laws for 99 and 101, it is seen, are very simple and readily applied.

CHAPTER VII.

PROPERTIES OF THE NUMBER NINE.

THE number Nine possesses the most remarkable properties of any of the natural numbers. Many of these properties have been known for centuries and have excited much interest among both mathematicians and ordinary scholars. So striking and peculiar are some of these properties that the number nine has been called "the most romantic" of all the numbers. On account of its relation to the numerical scale, if we get the factor 9 into a number it will cling to the expression and turn up in a variety of ways, now in one place and now in another, in a manner truly surprising. It reminds one of a mountain streamlet which ripples along its pathway, now buried beneath the ground and for awhile hidden from our sight, but presently gurgling to the surface at the most unexpected moment. It is no wonder that the property has been regarded as magical, and the number been called the "magical number." A few of these interesting properties will be here presented.

1. The first property of this number which attracts our attention is, that all through the column of "nine times" in the multiplication table, the sum of the terms is nine or a multiple of nine. Begin with twice nine, 18; add the digits together, and 1 and 8 are 9. Three times 9 are 27; 2 and 7 are 9. So it goes on up to eleven times nine, which gives 99. Add the digits; 9 and 9 are 18; 8 and 1 are nine. Go on in the same manner to any extent, and it is impossible to get rid of the figure 9. Multiply 326 by 9, and we have 2934, the sum of whose digits is 18, the sum of whose digits is 9. Let the

number nine once enter any calculation involving multiplication, and whatever you do, "like the body of Eugene Aram's victim," it is sure to turn up again. This curious property is explained by the principle of divisibility of numbers presented in the previous chapter. All these numbers being divisible by 9, the sums of their digits must be 9, or a multiple of 9.

2. Another curious property of the number nine is that if you take any row of figures and change their order as you please, the numbers thus obtained, when divided by 9, leave the same remainder. Thus, 42378, 24783, 82734, etc., when divided by 9 all give the same remainder, 6. The reason of this is, that the sum of the digits is the same, in whatever order they stand; and, as previously shown, the remainder from dividing a number by 9, is the same as the remainder from dividing the sum of its digits by 9.

3. An interesting principle is presented in the following puzzle, which, to the uninitiated, seems very singular. Take a number consisting of two places, invert the figures, and take the difference between the resulting number and the first number, and tell me one figure of the remainder and I will name the other. The secret is that the sum of the two digits of the remainder will always equal 9. Thus take 74, invert the terms, and we have 47; take the difference of the two numbers and we have 27, in which we see that the sum of 7 and 2 equals 9. In this case, suppose I had not known what number was taken; if the person had named one digit, say 2, I could have immediately named the other digit 7, since I know that the sum of the two digits is always 9.

The reason for this is that both numbers, having the same digits, are multiples of 9 with the same remainder; hence their difference is an exact multiple of 9, and consequently the sum of the two digits will equal 9. When the digits of the number are equal, the difference will be 0; and when they differ by unity, the difference will be 9.

4. There is another interesting puzzle, based upon these

principles, which is very curious to one who does not see the philosophy of it, and interesting to one who does. You tell a person to write a number of three or more figures; divide by 9, and name the remainder; erase one figure of the number; divide by 9, and tell you the remainder; and you will tell what figure was erased.

This is readily done when the principle is understood. If the second remainder is less than the first, the figure erased is the difference between the remainders; but if the second remainder is greater than the first, the figure erased equals the difference of the remainders subtracted from 9. The reason for this is that the remainder, after dividing a number by 9, is the same as the remainder after dividing the sum of the digits by 9, and hence the sum of the digits being diminished by the number erased, the remainder will also be diminished by it. If there is no remainder either time, then the term erased must be either 0 or 9.

To illustrate, suppose the number selected were 457; dividing by 9 the remainder is 7; erasing the second term and dividing, the remainder is 2; hence the term erased is 7 less 2 or 5. If the number were 461, dividing by 9, the remainder is 2; erasing the second term and dividing, the remainder is 5; hence the term erased must be the difference between 5 and 2, or 3, subtracted from 9, which is 6.

5. The following puzzle also arises from the principle of the divisibility by 9. Take any number, divide it by 9, and name the remainder; multiply the number taken by some number which I name, and divide the product by 9, and I will name the remainder. To tell the remainder, I multiply the first remainder by the number which I named as a multiplier, and divide this product by 9. The remainder thus arising will evidently be the same as the remainder which the person obtained.

6. If we take any number consisting of three consecutive digits and, by changing the place of the digits, make two other

numbers, the sum of these three numbers will be divisible by 9. This depends on the principle that the sum of any three consecutive digits is divisible by 3; and consequently each number, if not an exact multiple of 9, is a multiple of 9 plus 3, or of 9 plus a multiple of 3; and therefore the sum of three numbers is a multiple of 9 plus three 3's, and thus an exact multiple of 9. If we permutate the digits, making five other numbers, the sum of the six numbers will be divisible by twice 9; which may also be readily explained.

7. From the law of the divisibility by nine, several other properties, especially interesting to the young arithmetician, may be derived. Among these may be mentioned the following: 1. If we subtract the sum of the digits from any number the difference will be exactly divisible by 9. 2. If we take two numbers in which the sums of the digits are the same, the difference of the two numbers will be divisible by 9. 3. Arrange the terms of any number in whatever order we choose, and divide by 9, and the remainder in each case is the same. Such properties as these must have seemed exceedingly curious to the early arithmeticians, and fully entitle the number nine to be regarded as a magical number. All of these properties, it is proper to remark, would have belonged to the number eleven, if our scale had been duodecimal instead of decimal.

PART IV.

FRACTIONS.

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SECTION I.

COMMON FRACTIONS.

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DECIMAL FRACTIONS.

SECTION I.

COMMON FRACTIONS.

I. Nature of Fractions.

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III. Treatment of Common Fractions.

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CHAPTER I.

NATURE OF FRACTIONS.

THE Unit is the fundamental idea of arithmetic. From it arise two great classes of numbers—*Integers* and *Fractions.* Integers have their origin in the multiplication of the Unit; Fractions arise from the division of the Unit. One is the result of an immediate synthesis; the other, of a primary analysis. Fractions have their origin in the analysis of the Unit, as integers arise from the synthesis of units.

When the Unit is divided into equal parts, each part is seen to bear a certain relation to the Unit, and these parts may be collected together and numbered. This complex process of division, relation, and collection, gives us a fraction. The conception of a fraction, therefore, involves three things:—1st, a division of the unit; 2d, a comparison of the part with the unit; 3d, a collection of the equal parts considered. When a unit is divided into a number of equal parts, the comparison of the part with the unit gives the fractional idea, and the collection of the parts gives the fraction itself. Herein is clearly seen the distinction between an integer and a fraction. The former is an immediate synthesis; the latter involves a process of division, an idea of relation, and a synthesis of the parts. A fraction is, therefore, a triune product—a result of analysis, comparison, and synthesis.

Fractions, as has been stated, have their origin in a division of the Unit; they may also be derived from the comparison of numbers. Thus the comparison of *one* with *two*, or of *two* with *four*, may give the idea of *one-half;* and in a similar manner other fractions may be obtained. This, however, is a possible rather than the actual origin; fractions really originated in the division of the Unit.

When the Unit is divided into equal parts, these parts are collected and numbered as individual things; they may, therefore, be regarded as a special kind of *units*. To distinguish them from the Unit already considered, we call them *fractional units*. This gives us two classes of units, *integral units* and *fractional units*. The integral unit is known as the Unit; when fractional units are meant we use the distinguishing term *fractional*. The definite conception of an integer requires a clear idea of the Unit; the definite conception of a fraction requires a clear idea both of the integral and the fractional unit. The character of the thing divided, and the nature of the division, must be kept clearly before the mind, in order to obtain a distinct conception of the fraction. From this brief statement of the nature of the fraction we are prepared to define it.

Definition.—A fraction is a number of the equal parts of a Unit. This definition is an immediate inference from the conception of a fraction above presented. We divide the Unit into equal parts, and then take a *number* of these equal parts, and this is the fraction. A definition quite similar to this is, *a fraction is one or more of the equal parts of a unit.* This is not incorrect, though it is preferred to use the word "number" for "one or more." It is believed that the idea is thus expressed in the most concise and elegant form, and that it will meet the approval of mathematicians.

Several other definitions of a fraction have been presented by different authors, some of which are correct, while others are liable to serious objections. One writer says, "A fraction is a part of a unit." This is only part of the truth, for a fraction may be not only one part but several parts of a unit. Another writer says, "A fraction is an expression for one or more of the equal parts of a unit." In this definition the *expression*, the *written* or *printed symbols*, is made the fraction, which is evidently incorrect, as we have fractions previous to and independent of the expression of them. The expressions are not subjects of mathematical calculation, and hence they cannot be fractions. The same distinction holds between a fraction and its expression, as between a number and its expression. Thus we have the number four and the figure 4; so we have the fraction three-fourths, and the expression $\frac{3}{4}$, as two distinct things.

Another definition of a fraction is that it is an "unexecuted division." Says one writer, "A fraction is nothing more nor less than an unexecuted division." Says another, "A fraction may be regarded as an expression of an unexecuted division." This conception of a fraction is incorrect, as the idea of a fraction, and the idea of the division of one number by another, are entirely distinct. The fraction $\frac{4}{5}$ (4 fifths), means four of the equal parts which are obtained by dividing a unit into five equal parts. The division of 4 by 5 will give the expression $\frac{4}{5}$, but the idea of 4 divided by 5 is entirely distinct from the fractional idea; and hence the assertion, that a fraction is nothing more nor less than an unexecuted division, is absurd.

A fraction has also been defined to be the relation of a part of anything to the whole. This was the idea of Sir Isaac Newton, and is correct, though it is rather too abstract for a popular definition. Another form of stating the same idea is that "a fraction is that *definite part* which a portion is of the whole." Thus, if we divide an apple into two equal portions, one of these is *one-half* of the whole, and this definite part, *one-half*, is the fraction. This form of statement is not incorrect, though, like Newton's, it is too abstract for a popular definition.

Notation.—A fraction being a number of equal parts of a unit, it is natural that, in the notation of a fraction, we should indicate the number of parts used, by a figure. It would also, at first thought, seem natural to represent the name of the fractional unit by the words, *half*, *third*, etc., as 2 *thirds*, 3 *fourths*, etc.; or by their abbreviations, as 2-3ds, 3-4ths, etc. The letters would be finally omitted altogether, and the expressions become 2-3, 3-4, etc. This probably was the primary form, as is indicated by the expressions, 2-3 for 2 thirds, 3-4 for 3 fourths, which we meet in some of the older books.

It has been found more convenient, however, not to express directly the name of the part, but rather to represent the number of parts into which the unit is divided, from which the name of the part is inferred. This might have been done by writing one figure after another, 2-3; the 3 denoting the number of equal parts of the unit, and the 2 the number of parts considered. In practice it has been agreed, however, to write the figure denoting the number of parts into which the unit is divided, under the other, separating them by a line, as in division. The number expressed by the figure below the line is called the *denominator* of the fraction, the number expressed by the figure above the line is called the *numerator* of the fraction. The primary object of the figure below the line is not to name the fractional unit, but to denote the number of equal parts of a unit; from this the name of the fractional unit is inferred. Primarily, then, in our present notation, the *denominator* of the fraction is not the *denomination* of the fraction, though from the denominator the denomination is inferred. The denominator thus serves the double object of showing directly the number of equal parts into which the unit is divided, and, indirectly, the name or denomination of the fraction. This distinction should be carefully noted.

In integers we have one word to indicate the thing itself, and another to indicate the expression of it. Thus, *number* means the *how many*, or thing itself; and *figure*, the expression of it; the thing and its symbol being distinguished by independent names. In fractions there are no such terms to distinguish the expression of a fraction from the fraction itself. We are therefore obliged to use the same word *fraction* to designate both. This we are authorized to do by a figure of rhetoric called Metonymy, in which the name of an object is sometimes given to the symbol, or expression of the object. It is consequently allowable to use the word fraction when we mean the expression of a fraction, though this frequently occasions confusion and calls for particular care on the part of the teacher to prevent it. We are sometimes obliged to make the same dual use of the terms numerator and denominator, but should always do so with extreme caution to avoid confusion.

The expression of a fraction in its relation to the fraction itself, is seen, when analyzed, to be a more complicated thing than at first appears. To illustrate; first, we have the *fraction* itself, as so many *parts* of a unit; then we have the *two figures* to *represent* the fraction; and then we have the *numbers*, which these two figures denote; all of which should be carefully distinguished, if we would have a clear idea of the relation of a fraction to its notation. If we begin with the unit and compare it with the fraction as expressed, the matter becomes still more complicated. Thus, first we have the *Unit;* then the *equal parts* into which the Unit is divided; then the *relation* of these parts to the Unit; then the *expression* for a number of these parts, consisting of *two figures;* and then the *numbers* which these figures denote. It is therefore not entirely surprising that writers should have been careless and confused in their use of the terms relating to fractions.

History.—Before proceeding to the classification and treatment of Fractions, attention is called to what is known concerning their origin and history. The earliest work treating of the subject of Fractions is the *Lilawati* of the Hindoos. Fractions in the *Lilawati* are denoted by writing the numerator above the denominator, without any line between them. The introduction of the line of separation is due to the Arabs; and it is found in the earliest manuscripts on arithmetic. To denote a fraction of a fraction, as $\frac{2}{3}$ of $\frac{3}{4}$, the two fractions are written consecutively, without any symbol between them. To represent a number increased by a fraction, the fraction is written beneath the number; and when the fractiou is to be subtracted from the number a dot is prefixed to it; thus,

$$2\tfrac{1}{4} \text{ is denoted by } \begin{matrix}2\\ \tfrac{1}{4}\end{matrix} \text{ and } 3-\tfrac{1}{4} \text{ by } \begin{matrix}3\\ .\tfrac{1}{4}\end{matrix}$$

In other cases, their notation is not intelligible without verbal explanation, and the same is true of the Arabs and earlier European writers, who were singularly deficient in artifices of notation. In the solution of a problem in the *Lilawati*, in which "the fourth of a sixteenth of the fifth of three quarters of two-thirds of a moiety" is required, the work is written as indicated in the margin; which gives $\frac{6}{7680}$, or $\frac{1}{1280}$.

STATEMENT.

1	1	2	3	1	1	1
1	2	3	4	5	16	4

In solving the problem, "Tell me, dear woman, quickly, how much a fifth, a quarter, a third, a half and a sixth make when added together," the work appears in the *Lilawati* as indicated in the margin.

STATEMENT.

1	1	1	1	1	29
5	4	3	2	6	20

In solving the problem, "Tell me what is the residue of three, subtracting these fractions;" they expressed the work as indicated, which it is apparent could not be understood without an explanation.

STATEMENT.

3	1	1	1	1	1	31
1	5	4	3	2	6	20

The *Lilawati* contains four rules for the reduction and assimilation of fractions, as well as the application of their eight fundamental rules of arithmetic to them. These rules are clear and simple, and differ very little from those used in modern practice. That the author regarded fractions as somewhat difficult, is apparent from the following problem: "Tell me the result of dividing five by two and a third, and a sixth by a third, if thy understanding, sharpened into confidence, be competent to the division of fractions."

The notation of compound fractions varied with different authors; thus with Lucas di Borgo $\frac{2}{3}$ of $\frac{4}{5}$, or $\frac{2}{3}\times\frac{4}{5}$, was represented as in the margin, where v^a denotes *via*, or *times*. Stifelius denoted three-fourths of two-

$$\frac{2}{3}\;\overset{v^a}{\underset{v^a}{\overline{\underline{\quad\quad\quad}}}}\;\frac{4}{5}$$

thirds of one-seventh by writing the fractions nearly under one another as in the margin; and the same operation was indicated by Gemma Frisius thus:

$$\frac{3}{4}\ \Big|\ \frac{2}{3}\ \Big|\ \frac{1}{7}$$

$\frac{3}{4}$ $\frac{2}{3}$ $\frac{1}{7}$

a notation simple and convenient.

In the writings of Lucas di Borgo, when two fractions are to be added together or subtracted one from another, the operations to be performed are indicated as follows:

$$\begin{array}{l} 8 \quad 9 \\ \frac{2}{3} \times \frac{3}{4}\ \textit{fa}\ \frac{17}{12}\ \textit{vel.}\ \frac{1}{12}. \\ \ \ 12 \end{array}$$

where those quantities are to be multiplied together which are connected by the lines. There seems to be very little difference between the operations in fractions in ancient and modern text-books. In the works of Di Borgo and Tartaglia, the number of cases and their subdivisions are unnecessarily multiplied, and the reader is frequently more perplexed than instructed by the minuteness of their explanations. It may be remarked that the early writers seem to have been extremely embarrassed by the usage and meaning of the term multiplication in the case of fractions, where the product is less than the multiplicand; and some of their methods of explaining the seeming inconsistency are curious and ingenious.

CHAPTER II.

CLASSES OF COMMON FRACTIONS.

FRACTIONS are divided into two general classes—*Common* and *Decimal.* A *Common Fraction* is a number of equal parts of a unit, without any restriction as to the size of those parts. A *Decimal Fraction* is a number of the decimal divisions of a unit; that is, a number of *tenths, hundredths,* etc.

This distinction of fractions originated in a difference in the notation, rather than in any essential difference in the fractions themselves. It was seen that the decimal scale of notation, when extended to the right of the units place, was capable of expressing tenths, hundredths, etc., and that there would be a great advantage in such an expression of them; and thus the decimal fraction came to be regarded and treated as a distinct class. A brief discussion of each will be given.

Common Fractions are variously classified, according to different considerations. The primary division is that based upon their relative value compared with the Unit. Classifying them in reference to this relation, we have *Proper Fractions* and *Improper Fractions.* A Proper Fraction is one whose value is less than a unit; that is, one which is properly a fraction according to the primary conception of a fraction. An Improper Fraction is one which is equal to or greater than a unit; that is, one which is not properly a fraction in the primary meaning of the term.

Another division of common fractions arises from the idea of dividing a fraction into equal parts. A fraction originated in the division of the Unit into equal parts; now, if we extend this idea to obtaining a number of equal parts of a fraction, we get what is called a *Compound Fraction.* The Compound Fraction, it is thus seen, originated in the extension of the primary idea of division, which gave rise to the simple fraction. This idea of a compound fraction leads to the division of fractions into two classes—Simple Fractions and Compound Fractions. A Compound Fraction is technically defined as a fraction of a fraction.

If we extend the fractional idea a little further, and suppose the numerator, or denominator, or both, to become fractional, we have what arithmeticians call a *Complex Fraction.* The Complex Fraction may be defined as a fraction whose numerator, or denominator, or both, are fractional. Whether the complex fraction agrees with the definition of a fraction, or with the functions ascribed to the numerator or the denominator of a fraction, is a point which will be considered a little further on; but its origin was a natural outgrowth of the principle of pushing a notation to its limits. It should be noticed that the complex fraction may also have originated in the expression of the division of one fraction by another by writing the divisor under the dividend with a line between them; but the probabilities are that it originated as first indicated, by an extension of the fractional idea.

Fractions, therefore, are divided with regard to their *value*, as compared with the Unit, into Proper and Improper Fractions; with regard to their *form*, into Simple, Compound, and Complex. There is also another form of expressing fractional relations, so closely connected with the common fraction that it may be embraced under the same general head. I refer to the *Continued Fraction*, which will be treated with the general subject of common fractions.

Improper Fractions.—According to the primary idea, a fraction is regarded as a *part of a unit*, and hence as *less* than a unit. But since we can speak of any number of fractional units as we do of integral units, there arises a fractional expression whose value is greater than a unit. Thus we may speak of 5 *fourths*, 7 *fourths*, etc., although in a unit there are only 4 *fourths*. These we call improper fractions; that is they are improperly fractions from the primary idea of a fraction. The improper fraction presents several points of difficulty and interest, which will be briefly considered.

Take the expression $\$\frac{5}{4}$; is this strictly a fraction? That it is properly a fraction, appears from the definition of a fraction and from the discussion just given. How, then, shall it be read? If we read it "$\frac{5}{4}$ of a dollar," some one will object, that there are only four fourths in a dollar, and hence you cannot speak of five fourths of a dollar. If it be read "$\frac{5}{4}$ dollars," we will object, since there are not enough to make *dollars*, the plural meaning *two or more.* But, says some one, the grammars tell us that "the plural means more than one," and since $\$\frac{5}{4}$ is more than one, we may use the plural form and say "$\frac{5}{4}$ dollars." This, we reply, is a mere quibble, as the grammarians contemplate only integers when they say "more than one," and really mean "two or more." The reading "$\frac{5}{4}$ dollars" is, therefore, not strictly correct.

How, then, should it be read? I think the correct reading is "$\frac{5}{4}$ of a dollar." We mean by it five of such parts as are obtained by dividing a dollar into four equal parts. It is true there are not five fourths in one dollar, and the reading does not assume that there are. No one will object to saying $\frac{5}{4}$ of 100 cents equals 125 cents, which is equivalent to saying $\frac{5}{4}$ of a dollar equals a dollar and a quarter. The fractional units are *fourths of a dollar*, and the number of fractional units is *five;* hence the fraction is "five-fourths of a dollar." It is an improper fraction—improperly a fraction from the primary idea of a fraction—and in the name "improper fraction" we apparently enter a little protest against the absolute correctness of the reading in view of the primary idea of the fraction. If we have $\$\frac{8}{4}$ or $\$\frac{12}{4}$, we can then say $\frac{8}{4}$ dollars or $\frac{12}{4}$ dollars, since we then have "two or more." This discussion seems to have been called for from the fact that the question is often raised and debated as to what is the correct reading of the improper fraction.

Complex Fraction.—According to the strictest definition of a fraction, the complex fraction is an impossibility. This is rendered evident from a consideration of the functions ascribed to the denominator by the definition. The denominator shows the number of equal parts into which the unit is divided; hence, in the complex fraction $\frac{\frac{2}{3}}{\frac{3}{4}}$, the denominator, $\frac{3}{4}$, denotes that the unit is divided into $\frac{3}{4}$ equal parts. This is an impossibility, as may be seen at least in two ways. First, we can divide a unit into three or two equal parts, but not into one part, since there will be no division; and if we cannot divide it into one equal part, it is evident that we cannot divide it into less than one equal part. Secondly, if any one doubts the conclusion from this reasoning, let him take an apple and endeavor to divide it into $\frac{3}{4}$ equal parts. The effort I have sometimes known to be in a high degree amusing, and always conclusive of the correctness of the position assumed above.

A somewhat plausible argument in favor of the correctness of the complex fraction is the following: In the algebraic fraction $\frac{a}{b}$, the numerator and denominator are general expressions, and hence may represent fractions as well as integers. If then $b=\frac{3}{4}$ we shall have a complex fraction. This method of reasoning is too general for arithmetic; even in algebra it would prove that clearing the equation, $\frac{ax}{b}=\frac{c}{d}$, of fractions, does not clear it of fractions, since in $adx=bc$, each term may be a fraction. The expression $\frac{a}{b}$ means a divided by b, and is a fraction only so far as it coincides with our arithmetical idea of a fraction We conclude, therefore, that strictly speaking, the complex fraction is an impossibility. It is merely a convenient expression that one fraction is to be divided by another.

Should the idea and expression of a complex fraction, therefore, be discarded from arithmetic? This does not follow, and is not recommended. It is a convenient form of expressing the division of one fraction by another, and may thus be retained. Those who use it, however, should understand that it is not strictly a fraction, according to the primary idea of a fraction, but a representation of the division of a fraction by a fraction, or of a whole number and a fraction when only one term is fractional.

Is a Fraction a Number? It has been stated by some writers, and seems frequently to be the idea of pupils, that a fraction is not a number. This, however, is a mistake, as will appear from a slight consideration of the matter. Newton's definition of number provides for the fractional number when the object measured is a definite part of the measure; it consequently appears that the fraction is a number, if we accept his definition as correct. The definition, "A Fraction is a *number* of equal parts of unity," also makes it clear that a fraction is a number. Again, if it is not a number, what kind of a quantity is it; and why should it be treated in arithmetic, the science of numbers? *Five inches* is certainly a number; hence its equivalent, *five-twelfths* of a foot, is also a number. Numbers are of two classes, *integers* and *fractions;* and fractions are numbers, as much so as integers. The fractional number, it will be noticed, involves two ideas—first, the integral unit; and second, the fractional unit. In an integer we have the idea of a number of units; in the fraction we have, not only an idea of a number of units, but also the *relation* of the fractional unit to the integral unit.

Is a Fraction a Denominate Number? It has been affirmed by some authors that "fractions are a species of denominate numbers." This, however, is true only in a very limited or partial sense. *Three quarts* is not precisely the same as *three-fourths of a gallon,* though they are equal in value. In the latter case, there is a direct and necessary relation of a part to a unit; in the former case, no such relation is implied. To understand the fraction, *three-fourths* of a gallon, the idea of the unit, gallon, must be in the mind; in *three quarts* no such condition is necessary. In one case there are two units considered, the *gallon* and the *fourth;* in the other case but one unit, the *quart,*—not considering the unit of the pure numbers, *three* and *four* themselves. *Fourths* have reference to the integral unit, and always imply this relation; *quarts* have no reference to gallons, and do not imply gallons.

Again, the fraction *three-fourths* may be used entirely distinct from any denominate unit, and in this case it must be an abstract, not a denominate number. Two is *one-fourth* of eight; here the measure of this relation, *one-fourth,* cannot but be abstract. It is evident, therefore, that a fraction is not a denominate number. There are abstract and denominate fractions, as there are abstract and denominate integers.

CHAPTER III.

TREATMENT OF COMMON FRACTIONS.

A FRACTION has been defined as a number of the equal parts of a unit. The parts into which the unit is divided are called *fractional units*. A fraction may, therefore, also be defined as a number of fractional units. Fractions are divided, as previously stated, into Common and Decimal Fractions.

A Common Fraction is a number of fractional units expressed with a numerator and a denominator; as two-thirds, written $\frac{2}{3}$. The denominator of a fraction denotes the number of equal parts into which the unit is divided. The numerator of a fraction denotes the number of fractional units in the fraction. A common fraction is usually expressed by writing the numerator above the denominator with a line between them. Care should be taken not to define the denominator as the "*figure* below the line," and the numerator as the "*figure* above the line;" and then speak of multiplying the numerator and denominator. This will lead one to suppose that *figures* may be multiplied, rather than the numbers which they represent. It is surprising that so many writers upon arithmetic should have fallen into this error.

Cases.—Fractions admit of the same general treatment as integers; we therefore have the same fundamental cases in fractions as in whole numbers. These cases are all embraced under the general processes of *Synthesis*, *Analysis*, and *Comparison.* The cases of synthesis and analysis are the same as in whole numbers. To perform the synthetic and analytic processes, we need to change fractions from one form to another;

hence *Reduction* enters largely into the treatment of fractions. The comparison of fractions gives rise to several cases called the *Relation of Fractions*, which do not appear in whole numbers. The various cases of fractions then are; Reduction, Addition, Subtraction, Multiplication, Division, Relation, Composition, Factoring, Common Divisor, Common Multiple, Involution, and Evolution.

A complete view of the fundamental processes is presented in the following logical outline. Composition, Factoring, Involution, and Evolution, presenting no points different from those of whole numbers, are omitted in the treatment. The other cases arising out of Comparison apply equally to integers and fractions, and do not require a distinct treatment.

Outline of the Cases of Fractions.

- 1. REDUCTION.
  - 1. Number to a Fraction.
  - 2. Fraction to a Number.
  - 3. To Higher Terms.
  - 4. To Lower Terms.
  - 5. Compound to Simple.
  - 6. Dissimilar to Similar.
- 2. ADDITION.
  - 1. The denominators alike.
  - 2. The denominators unlike.
- 3. SUBTRACTION.
  - 1. The denominators alike.
  - 2. The denominators unlike.
- 4. MULTIPLICATION.
  - 1. Fraction by a Number.
  - 2. Number by a Fraction.
  - 3. Fraction by a Fraction.
- 5. DIVISION.
  - 1. Fraction by a Number.
  - 2. Number by a Fraction.
  - 3. Fraction by a Fraction.
- 6. RELATION.
  - 1. Number to a Number.
  - 2. Fraction to a Number.
  - 3. Number to a Fraction.
  - 4. Fraction to a Fraction.

The "Relation of Fractions" is a new division of the subject of fractions: it was first published in the Normal Written Arithmetic, in 1863, and has since been introduced into several other works on written arithmetic, and will probably be generally adopted.

Methods of Treatment.—There are two methods of developing the subject of common fractions, which may be distinguished as the *Inductive* and *Deductive* methods. These two methods are entirely distinct in principle and form; and the distinction, being new, seems worthy of special attention.

By the Inductive Method, we solve each case by *analysis*, and derive the *rules*, or *methods of operation*, from these analyses, by inference or induction. The method is called *inductive*, because it proceeds from the analysis of particular problems to a general method which applies to all problems of a given class. The solutions, it will be noticed, are independent of any previously established principles of fractions, each case being treated by the method of arithmetical analysis which reasons to and from the Unit.

To illustrate the method we will take the problem, "In $\frac{3}{4}$ how many twentieths?" We analyze this as follows: *One* equals $\frac{20}{20}$, and $\frac{1}{4}$ equals $\frac{1}{4}$ of 20 *twentieths*, or 5 twentieths; and $\frac{3}{4}$ equals 3 times 5 twentieths, or 15 twentieths; hence $\frac{3}{4}$ equals $\frac{15}{20}$. Now, by examining this solution, we see that we multiply the numerator of $\frac{3}{4}$ by the number which denotes how many times *four*, the given denominator, equals the required denominator, *twenty*, which is the same as multiplying both terms of $\frac{3}{4}$ by the same number, *five;* hence we derive the *rule*, "to reduce a fraction to higher terms, multiply both terms by the same number."

For another illustration, take the converse of this problem, "In $\frac{15}{20}$ how many fourths?" The solution is as follows: *One* equals $\frac{20}{20}$, and $\frac{1}{4}$ equals $\frac{1}{4}$ of $\frac{20}{20}$, which is $\frac{5}{20}$; hence $\frac{1}{5}$ of the number of 20ths equals the number of 4ths; $\frac{1}{5}$ of 15 is 3, hence $\frac{15}{20}$ equals $\frac{3}{4}$. This is the analysis of the problem; we then proceed to derive a rule by which all such problems may be solved. By examining this analysis, we see that we take the same part of the numerator for the numerator of the required fraction that the denominator of the required fraction is of the denominator of the given fraction; hence we derive the rule, "to reduce a fraction to lower terms divide both numerator and denominator by the same number." This rule is thus obtained by analyzing the analysis; it may also be obtained by comparing the two fractions. Thus, comparing $\frac{3}{4}$ and $\frac{15}{20}$, we see that 3 equals 15 divided by 5, and 4 equals 20 divided by 5—that is, both divided by the same number—and seeing that this principle holds good in several cases, we infer the rule.

By the Deductive Method we first establish a few general principles by demonstration, and then derive the *rules*, or methods of operation, from these principles. The method is called *deductive* because it proceeds from the general principle to the particular problem. To illustrate this method, let us solve the same problem, "Reduce $\frac{3}{4}$ to twentieths." By a general proposition which we assume has been demonstrated, we have the principle, "Multiplying both terms of a fraction by any number does not change its value;" hence we may reduce $\frac{3}{4}$ to twentieths by multiplying both terms by 5, which will give the required denominator, and we have $\frac{3}{4}$ equal to $\frac{15}{20}$.

For another illustration, we will solve the converse problem, "Reduce $\frac{15}{20}$ to fourths." By a general proposition, which we assume has been demonstrated, we have the principle, "Dividing both terms of a fraction by the same number does not change its value;" hence we may reduce $\frac{15}{20}$ to fourths by dividing both numerator and denominator by any number which will give the required denominator. This number, we see, is 5; hence, dividing both numerator and denominator by 5, we have $\frac{15}{20}$ equal to $\frac{3}{4}$.

We will illustrate the difference of these two methods still further by a problem in compound fractions. Take the question, "What is $\frac{2}{3}$ of $\frac{4}{5}$?" The analysis is as follows: $\frac{1}{3}$ of $\frac{1}{5}$ is one of the three equal parts into which $\frac{1}{5}$ may be divided; if each 5th is divided into 3 equal parts, $\frac{5}{5}$ or the Unit will be divided into 5 times 3, or 15 equal parts, and each part will be $\frac{1}{15}$; hence $\frac{1}{3}$ of $\frac{1}{5}$ is $\frac{1}{15}$, and $\frac{1}{3}$ of $\frac{4}{5}$ is 4 times $\frac{1}{15}$, or $\frac{4}{15}$, and $\frac{2}{3}$ of $\frac{4}{5}$ is 2 times $\frac{4}{15}$, or $\frac{8}{15}$. Examining this analysis, we see that we have multiplied the two denominators together and the two numerators together, from which we derive the rule for the reduction of compound fractions. By the deductive method we would reason as follows: By a principle previously demonstrated, $\frac{1}{3}$ of $\frac{4}{5}$, which is the same as dividing $\frac{4}{5}$ by 3, is $\frac{4}{15}$; and $\frac{2}{3}$ of $\frac{4}{5}$ by another principle, is $\frac{8}{15}$. It will be noticed that the deductive method is much shorter than the inductive method, because while the former explains every point involved, the latter makes use of principles previously demonstrated. If in the deductive solution, we should stop and demonstrate the principles we are to use, it would make the solution much longer. The difference of the two methods can also be clearly illustrated in the division and relation of fractions. In my higher arithmetic the two methods are presented in each case, where a full comparison may be made of them.

The distinction between these two methods is broad and emphatic. By the Inductive Method the problem is solved without any reference to any previously established principle; by the Deductive Method, the solution is derived from a general principle supposed to have been previously demonstrated. Both of these methods may be used in the development of fractions, and it is a question worthy of consideration which is to be preferred.

The Inductive Method is believed to be simpler and more easily understood by young pupils. It is especially adapted to beginners, since it proceeds according to the simple steps of analysis, or the comparison of the collection with the unit. It also follows the law of the development of the young mind—"from the particular to the general." It is especially suited to the subject of Mental Arithmetic, on account of its simplicity and the mental discipline it is calculated to afford.

The Deductive Method is more difficult in thought than the Inductive Method. Young pupils always find a difficulty in founding a process of reasoning upon previously established principles. It is not natural for the youthful mind to reason from generals to particulars. Besides, the demonstrations of these general principles are not readily understood by young pupils. With much experience as a teacher, I state that it is a rare thing to find a pupil who can give a good logical demonstration of these principles, and text-books and teachers often do no better. The so-called demonstrations in many of our text-books are mere explanations or illustrations, and not logical proofs of the propositions. To say that "multiplying the denominator of a fraction increases the number of parts of the fraction, and diminishes their size in the same proportion," is a loose sort of statement that comes very far short of scientific demonstration. We will consider these principles and their demonstration.

Fundamental Principles.—In the Deductive Method, we have stated, we first establish several general principles, and then derive the rules or methods of operation from them. These principles relate to the multiplication of the numerator and denominator of a fraction. They may be demonstrated in two distinct ways. One of these is founded upon the principles of division; the other upon the nature of the fraction and the functions of the numerator and denominator. All the various methods in our text-books on arithmetic may be embraced under these two general methods.

The Method of Division is employed by a large majority of our writers on arithmetic. This method consists in regarding the fraction as an expression of an unexecuted division, the numerator representing the dividend, and the denominator the divisor, and the value of the fraction being the quotient. Then, by principles of division presumed to have been previously established, since dividing the *dividend* divides the *quotient*, dividing the *numerator* divides the *fraction;* and since multiplying the *divisor* divides the *quotient*, multiplying the *denominator* divides the *fraction*, etc.

The Fractional Method of demonstrating these fundamental principles is based upon the nature of the fraction itself. It regards the fraction as a number of equal parts of a unit, and determines the result of these operations by comparing the fractional unit with the Unit. Thus, if we multiply the denominator of a fraction by any number, as three, the Unit will be divided into *three* times as many equal parts, hence each part will be *one-third* as large as before; and the same number of parts being taken, the value of the fraction will be *one-third* as large as before. In a similar manner all the principles may be demonstrated.

The Fractional Method is undoubtedly the correct one. The Method of Division is liable to several objections, and should be discarded in teaching and in writing text-books, as appears from several considerations.

First, it is illogical to leave the conception of a fraction and pass to that of division, to establish a principle of a fraction. A fraction and an expression of division are two distinct things, and should not be confounded. The fraction $\frac{3}{4}$ is *three-fourths*, and does not mean 3 divided by 4. It is true that the *expression* $\frac{3}{4}$ does also mean 3 divided by 4; but when we regard it as a fraction we have and should have no idea of the division of three by four. It is, therefore, illogical, I say, to convert a fraction into a division of one number by another to attain to a principle of the fraction.

Secondly, it is not only illogical to treat the subject in this manner, but it does not give the learner the true idea of it. He may see that multiplying the denominator does divide the value of the fraction, but he will not see down into the core of the matter, *why* it does so. The method, to say the least, gives but a superficial view of the subject, and is therefore objectionable. If the fraction will admit of a simple treatment as a fraction, it is absurd to transform it into something else to prove its principles.

It may be said in favor of the method of division, that it is simpler and more easily understood by the learner; but this both theory and experience in instruction will disprove. I believe that the pupil can quite as readily see that dividing the numerator of a fraction divides the value of the fraction, as he can see that dividing the dividend divides the quotient; and the same holds for the other principles. This method may sometimes seem a little easier to the learner, because it depends upon an assumed principle; but require the pupil to prove that principle, and he will find it quite as difficult as to prove the fractional principle itself. For the method of demonstrating these theorems which the author prefers in arithmetic, the reader is referred to his arithmetical works.

CHAPTER IV.

CONTINUED FRACTIONS.

EVERY new idea, when once fixed, becomes a starting point from which we pass to other new ideas. The mind never rests satisfied with the old; it is always reaching out beyond the known into the unknown. "Still sighs the world for something new," is as true in science as in society. Given a new conception, and the tendency is to push it forward until it leads us to other ideas and truths not anticipated in the original conception. Thus, from the original idea of a simple fraction originated the compound and complex fractions; and thus also by extending the original conception, arose the *Continued Fraction.*

Definition.—A *Continued Fraction* is a fraction whose numerator is 1, and denominator an integer plus a fraction whose numerator is also 1 and denominator a similar fraction, and so on. Thus,

$$\tfrac{49}{155} = \cfrac{1}{3 + \cfrac{1}{6 + \frac{1}{8}}}$$

Several recent authors, for convenience, write a continued fraction with the sign of addition between the denominators; thus, $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$.

Origin.—Continued Fractions were first suggested to the world in a work by Cataldi, published in 1613, at Bologna. Cataldi reduces the square roots of even numbers to continued fractions, and then uses these fractions in approximation, though without the modern rule by which each approximation is educed from the preceding two. Continued Fractions were also proposed about the year 1670, by Lord Brouncker, President of the Royal Society. It is known that in order to express the ratio of the circumscribed square to the circumference of the circle, he derived the following continued fraction given in the margin; but by what means he was led to it, has not been ascertained. Dr. Wallis subsequently added to and improved the subject, giving a general method of reducing all kinds of continued fractions to common fractions.

$$1+\frac{1}{2}+\frac{9}{2}+\frac{25}{2}+\text{etc.}$$

The complete development of these fractions, with their application to the solution of numerical equations and problems in indeterminate analysis, is due to the Continental mathematicians. Huygens is said to have explained the manner of forming the fractions by continual divisions, and to have demonstrated the principal properties of the converging fractions which result from them. John Bernoulli made a happy and useful application of the continued fraction to a new species of calculation which he devised for facilitating the construction of tables of proportional parts.

Treatment.—The subject of continued fractions is most conveniently treated by the algebraic method, and may be found quite fully presented in some of the works on higher algebra. In this place we shall briefly consider: 1. Reducing common fractions to continued fractions; 2. Reducing continued fractions to common fractions; 3. Their application; 4. Their principles.

We shall first show how a common fraction may be reduced to a continued fraction. Take the common fraction $\frac{68}{157}$. Dividing both numerator and denominator by 68, we have the first expression in the margin; dividing the numerator and denominator of the second fraction by 21, we have the second expression in the margin; dividing again by 5, we have the third expression in the margin; which finishes the division, as

$$\frac{1}{2}+\frac{21}{68} \qquad \frac{1}{2}+\frac{1}{3}+\frac{5}{21}$$

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$$

the numerator of the last fraction is unity. The terms $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., are called the first, second, third, etc., partial fractions.

The same result may be obtained by dividing as in finding the greatest common divisor, and taking the several quotients for the successive denominators. Taking $\frac{68}{157}$, and dividing as if to find the greatest common divisor of its terms, we see that the resulting quotients are the same as the denominators of the partial fractions. Hence we derive the following rule for reducing common fractions to continued fractions: *Find the greatest common divisor of the terms of the given fraction; the reciprocals of the successive quotients will be the partial fractions which constitute the continued fraction required.*

68	157	
63	136	2
5	21	3
5	20	4
	1	5

Let us now see how a continued fraction may be reduced to a common fraction. This reduction may be effected in two ways; by beginning at the last fraction and working up, or by beginning at the first fraction and working down.

If we take the continued fraction given in the margin and reduce the complex fraction formed by the last two terms to a simple fraction, we shall have $\frac{5}{21}$. Taking this result and the preceding partial fraction together,

$$\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{4+}\frac{1}{5}$$

we have $\frac{1}{2+}\frac{5}{21}$, which reduced equals $\frac{21}{47}$. Joining this to the preceding term, we have $\frac{1}{1+}\frac{25}{47}$, which equals $\frac{47}{68}$. Finally, $\frac{1}{3+}\frac{47}{68}=\frac{68}{251}$, the value of the fraction.

By beginning at the first fraction, approximate values of the continued fraction may be obtained by respectively reducing two, three, or more of the partial fractions to simple fractions. Thus, in the fraction given above, the first approximate value is $\frac{1}{3}$; the second $\frac{1}{3+}\frac{1}{1}$, or $\frac{1}{4}$; the third is $\frac{1}{3+}\frac{1}{1+}\frac{1}{2}$, or $\frac{3}{11}$; the fourth $\frac{13}{48}$; the fifth $\frac{68}{251}$.

By exhibiting this process in an analytic form, a law may be discovered which presents a simpler and easier method of finding approximate values than either of the others.

Let us take the fraction in the margin and find its successive approximate values, and notice the law of the derivation of one approximation from the previous ones. The work may be written as follows:

$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{4}}}}$

$$\frac{1}{2} = \frac{1}{2}, \text{ 1st approx. val.}$$

$$\cfrac{1}{2+\cfrac{1}{3}} = \frac{3}{3\times2+1} = \frac{3}{7}, \text{ 2d " "}$$

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5}}} = \cfrac{1}{2+\cfrac{5}{3\times5+1}} = \frac{3\times5+1}{(3\times2+1)\times5+2} = \frac{3\times5+1}{7\times5+2} = \frac{16}{37}, \text{ 3d " "}$$

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{4}}}} = \cfrac{1}{2+\cfrac{5+\frac{1}{4}}{3\times(5+\frac{1}{4})+1}} = \frac{(3\times5+1)\times4+3}{\{(3\times2+1)\times5+2\}\times4+3\times2+1} = \frac{16\times4+3}{37\times4+7} = \frac{67}{155}.$$

We take $\frac{1}{2}$, the first term of the continued fraction, for the first approximate value. Reducing the complex fraction formed by the first two terms of the continued fraction, we have $\frac{3}{7}$ for the second approximate value. Continuing the reduction, we obtain $\frac{16}{37}$ and $\frac{67}{155}$ for the remaining values. Examining the last two reductions, we find that the third approximate value is obtained by multiplying the terms of the second approximate fraction by the denominator of the third partial fraction, and adding to these products the corresponding terms of the first approximate fraction. We see also that the fourth approximate value is equal to the product of the terms of the third approximate value by the denominator of the fourth partial fraction, plus the corresponding terms of the second approximation. Hence we derive the following rule:

For the first approximate value take the first partial fraction; for the second value, reduce the complex fraction formed by the first two terms of the continued fraction; for each succeeding approximate value, multiply both terms of *the approximation last obtained by the next denominator of the continued fraction, and add to the products the corresponding terms of the preceding approximation.*

We will now show the application of continued fractions by the solution of several practical questions.

1. Let it be required to express approximately, in the fraction of a day, the difference between a solar year and 365 days.

By the old reckoning, the excess of the solar year over 365 days was 5 hours, 48 minutes, 48 seconds. Reducing, we find this excess equals 20,928 seconds, and 24 hours equals 86,400 seconds. Therefore, the true value of the fraction $=\frac{20928}{86400}=\frac{109}{450}$. Now, converting $\frac{109}{450}$ into a continued fraction, we have the expression given in the margin, from which, by the last rule, we obtain the approximate values $\frac{1}{4}$, $\frac{7}{29}$, $\frac{8}{33}$, $\frac{31}{128}$, $\frac{39}{161}$, $\frac{109}{450}$.

$$\frac{109}{450}=\cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2}}}}}}$$

The fraction $\frac{1}{4}$ agrees with the correction introduced into the calendar by Julius Cæsar, by means of *bissextile or leap year*. The fraction $\frac{8}{33}$ is the correction used by the Persian astronomers, who add 8 days in every 33 years, by having 7 regular leap-years, and then deferring the eighth for 5 years.

2. Required the approximate ratio of the English foot to the French metre containing 39.371 inches.

The true ratio is $\frac{12000}{39371}$. Reducing to a continued fraction, we find some of the first approximate values to be $\frac{1}{3}$, $\frac{3}{10}$, $\frac{4}{13}$, $\frac{7}{23}$, $\frac{25}{82}$, $\frac{32}{105}$. Hence the foot is to the metre as 3 to 10, nearly; a more correct ratio is 32 to 105.

3. To find some of the approximate values of the ratio of the circumference of a circle to the diameter.

Taking the value of the circumference of the circle whose diameter is 1, to 10 places of decimals, the ratio of the diameter to the circumference will be expressed by the common fraction $\frac{10000000000}{31415926535}$. Reducing to a continued fraction, some of the first approximate values are, $\frac{1}{3}$, $\frac{7}{22}$, $\frac{106}{333}$, $\frac{113}{355}$. Inverting these fractions, we have the ratio of the circumference to the diameter, which is the ratio commonly used. The second gives $\frac{22}{7}$ which is the ratio said to have been found by Archimedes; and the fourth gives $\frac{355}{113}$, which is the same as that determined by Metius, which is more exact than 3.141592, from which it is derived.

Continued fractions have been employed for obtaining elegant approximations to the roots of surds. Thus, let it be required to find the square root of $\frac{1}{2}$, or the ratio of the side of a square to its diagonal.

The square root of $\frac{1}{2}$, or $\sqrt{\frac{1}{2}}$, equals $\frac{1}{\sqrt{2}}$. Dividing both terms by the numerator we have $\frac{1}{\sqrt{2}}=\cfrac{1}{1+\cfrac{\sqrt{2}-1}{1}}$. Multiplying both terms of the fraction $\frac{\sqrt{2}-1}{1}$ by $\sqrt{2}+1$, it will become $\frac{1}{\sqrt{2}+1}=\cfrac{1}{2+\cfrac{\sqrt{2}-1}{1}}$. Substituting, we have

$$\frac{1}{\sqrt{2}}=\cfrac{1}{1+\cfrac{1}{2+\cfrac{\sqrt{2}-1}{1}}}.$$

Again, the fraction $\frac{\sqrt{2}-1}{1}$ becomes, as before, equal to $\cfrac{1}{2+\cfrac{\sqrt{2}-1}{1}}$ and by thus continuing the process, we find $\frac{1}{\sqrt{2}}$ to equal the following continued fraction:

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\text{etc.}}}}}}$$

Some of the first approximate values of this fraction are $\frac{1}{1}$, $\frac{2}{3}$, $\frac{5}{7}$, $\frac{12}{17}$, $\frac{29}{41}$, $\frac{70}{99}$, $\frac{169}{289}$, etc.

Continued fractions are also applied to the solution of indeterminate problems, as may be seen in Barlow's *Theory of Numbers*, or Legendre's *Théorie des Nombres*.

There are several beautiful principles belonging to the approximate values of continued fractions, a few of which we present in this place. The values just obtained for the ratio of the side of a square to its diagonal are used as illustrations.

1. *The approximate fractions are alternately too small and too large.* Thus, $\frac{2}{3}$, $\frac{12}{17}$, $\frac{70}{99}$, are too small, while $\frac{1}{1}$, $\frac{5}{7}$, $\frac{29}{41}$, and $\frac{169}{239}$ are too large.

2. *Any one of these fractions differs from the true value of the continued fraction by a quantity which is less than the reciprocal of the square of its denominator.* Thus, $\frac{12}{17}$, which is the ratio much used by carpenters in cutting braces, differs from the true ratio by a quantity less than $(\frac{1}{17})^2=\frac{1}{289}$.

3. *Any two consecutive approximate fractions, when reduced to a common denominator, will differ by a unit in their numerators.* Thus $\frac{5}{7}$ and $\frac{12}{17}$, when reduced to a common denominator, become $\frac{85}{119}$ and $\frac{84}{119}$.

4. *All approximate fractions are in their lowest terms.* If they were not, the difference of the numerators of two consecutive approximate fractions, when reduced to a common denominator, would differ by more than unity. For each numerator is multiplied by the denominator of the other fraction, hence one derived numerator contains the original numerator, and the other the original denominator of either fraction. If then there were a common factor, it must be a factor of the difference of the numerators; and this difference would be greater than unity, which is contrary to the previous principle.

The successive approximate values are called the *convergents* of the fraction. The numerator or denominator of the convergent is called, by Sylvester, a *cumulant.* A non-terminating continued fraction whose quotients recur, is called a *periodical* or *recurring* continued fraction. Its value can be shown to be equal to one of the roots of a quadratic equation. It can also be shown that every quadratic surd gives rise to an equivalent periodic continued fraction.

SECTION II.

DECIMAL FRACTIONS.

I. Origin of Decimals.

II. Treatment of Decimals.

III. Nature of Circulates.

IV. Treatment of Circulates.

V. Principles of Circulates.

VI. Complementary Repetends.

VII. A New Circulate Form.

CHAPTER I

ORIGIN OF DECIMALS.

THE invention of the Decimal Fraction, like the invention of the Arabic scale, was one of the happy strokes of genius. The common fraction was expressed by a notation quite distinct from that of integers, and required not only a different treatment, but one much more complicated and difficult. The expression of the decimal divisions of the unit in the same scale with integers, and the possibility of reducing common fractions to the decimal form, wrought quite a revolution in the science of arithmetic, and has greatly simplified it. This new method of expressing fractions gave rise to a much simpler method of treating them; and has elevated the decimal fraction into distinction, and gained for it an independent consideration.

Origin.—The Decimal Fraction had its origin within the last three centuries. Theoretically it may have originated in either of two ways. There may have been a transition from the common fraction to the decimal, by noticing that a number of tenths, hundredths, etc., might be expressed by the decimal scale. This is the manner in which the subject is usually presented in the text-books of the present day. Thus, after the pupil is made familiar with the fractions $\frac{1}{10}$, $\frac{1}{100}$, etc., it is stated that $\frac{1}{10}$ may be expressed thus, .1; $\frac{1}{100}$ thus, .01, etc. The decimal fraction could also have arisen directly from the decimal scale. Thus, since the law of the scale is, that terms diminish in value from left to right in a ten-fold ratio, the idea of carrying the scale on to the right of the unit would naturally present itself, and such a continuation would give rise to the decimal. As the unit was one-tenth of the tens, the first place to the right of the unit would be one-tenth of the unit; the second place, one-tenth of one-tenth, or one-hundredth, etc. These two methods of conceiving the origin of decimals are entirely distinct; indeed, they are the converse of each other. In one case we pass from the common fraction to its expression in the decimal scale; in the other we pass from the expression in the decimal scale to the fraction. This distinction, it may be remarked, has a practical bearing upon the method of teaching the subject. In which way it did actually originate is not definitely known, though De Morgan holds that the table of compound interest suggested decimal fractions to Stevinus.

History.—The introduction of decimal fractions was formerly ascribed to Regiomontanus, but subsequent investigations have shown this to be incorrect. The mistake seems to have arisen from the confused manner in which Wallis stated that Regiomontanus introduced the decimal radius into trigonometry in place of the sexagesimal. Decimal fractions were introduced so gradually that it is difficult, if not impossible, to assign their origin to any one person. The earliest indications of the decimal idea are found in a work published in 1525 by a French mathematician named Orontius Fineus. In extracting the square root of 10, he extracts the approximate root of 10000000 and obtains 3162. He then separates 162, which with him *is not a fraction*, but only a means of *procuring fractions*, and converts it, after the scientific custom of the times, into sexagesimal fractions (having as base 60), so that the square root of 10 would be expressed $3\ 9'\ 43''\ 12'''$, or $3+\frac{9}{60}+\frac{43}{3600}+\frac{12}{216000}$. He concludes that chapter of his book by stating that in this 162, 1 is a tenth, 6 is six hundredths, etc., so that it would seem that he had quite a clear notion of decimals.

Tartaglia, in 1556, gives a full account of the method of Orontius, but prefers the common fractional form $3\frac{162}{1000}$. In Recorde's *Whetstone of Witte*, 1557, the same rule is copied; but after obtaining three decimal places of the square root, the remainder is written as a common fraction. Peter Ramus, in an arithmetic published in Paris in 1584 or 1592, also quotes the rule of Orontius.

In 1585, Stevinus wrote a special treatise in French, called "The DISME, *by the which we can operate with whole numbers without fractions.*" It was first published in Dutch about the year 1590, and describes in very express and simple terms the advantages to be derived from this new arithmetic. Decimals are called *nombres de disme:* those in the first place whose sign is (1) are called *primes,* those in the second place whose sign is (2) are called *seconds,* and so on; whilst all integers are characterized by the sign (0), which is put over the last digit. The following are some of his arithmetical operations by means of decimals, representing multiplication and division.

```
           (0)(1)(2)
          3  2  5  7
          8  9  4  6
      ---------------
       1  9  5  4  2
    1  3  0  2  8
 2  9  3  1  3
 2  6  0  5  6
---------------------
 2  9  1  3  7  1  2  2
```

```
 (0)(1)(2)(3)(4)(5)(1)(2)
  3  4  4  3  5  2 (9  6

     1
     1  8  6
     5  1  1  4
     7  6  3  7          (0)(1)(2)(3)
  3  4  4  3  5  2 (3  5  8  7
     9  6  6  6  6
        9  9  9
```

It will be seen that he employs the "scratch method" of division. The following is an example of indefinite division found in his work:

```
     (0)(1)(2)(3)
4/3 = 1  3  3  3
```

In this treatise Stevinus proposed to supersede fractions by *dismes,* or decimals. He enumerates the advantages which would result from the decimal subdivision of the units of length, area, capacity, value, and lastly of a degree of the quadrant, in the uniformity of notation, and the increased facility of performing all arithmetical operations in which fractions of such units were involved. It is remarkable, however, that though while he confines himself to the *matter* of his computation he admits his *dismes*, when he passes to their *form* he converts them into integers. Still, he must be regarded as the real inventor and introducer of the system of decimals. De Morgan says "The *Disme* is the first announcement of the use of decimal fractions;" and Dr. Peacock also remarks that "the first notice of decimals, properly so called, is to be found in *La Disme*."

This work of Stevinus was translated into English in 1608, by Richard Norton, under the title, "*Disme*, the arte of tenths, or decimal Arithmetike, teaching how to perform all computations whatsoever by whole numbers without fractions, by the four principles of common Arithmetike: namely, addition, subtraction, multiplication, and division, invented by the excellent mathematician, Simon Stevin." In this work the notation is changed to

$$\overset{(1)}{3}, \overset{(2)}{7}, \overset{(3)}{5}, \overset{(4)}{9}.$$

The introduction of decimals into works on arithmetic was slow, even after their use had been shown by Stevinus. One of the earliest English works in which decimal fractions are really used, is that of Richard Witt, 1613, containing tables of half-yearly and compound interest. These tables are constructed for ten million pounds; seven figures are cut off, and the reduction to shillings and pence, with a *temporary* decimal separation, is introduced when wanted. Thus, when the quarterly table of amounts of interest at ten per cent. is used for three years, the principal being 100*l.*, in the table stands 1372-66429, which multiplied by 100 and seven places cut off, gives the first line of the following citation:

"The Worke

Facit	l	1372	66429
	sh	13	2858
	d	3	4296."

Giving 1372*l.* 13*s.* 3*d.* for the answer. The tables are expressly stated to consist of *numerators*, with 100... for a denominator. Napier's work, published in 1617, contains a treatise on decimals, though he does not use the decimal point, except in one or two instances, but rather indicates the place of the decimal figures by primes, seconds, etc., according to the method of Stevinus. The author expressly attributes the origin of decimals to Stevinus.

In 1619 we find the contents of Norton's treatise embodied in an English work entitled, "The Art of Tens, or Decimall Arithmetike, wherein the art of Arithmetike is taught in a more exact and perfect method, avoyding the intricacies of fractions. Exercised by Henry Lyte, Gentleman, and by him set forth for his countries good. London, 1619." It is dedicated to Charles, Prince of Wales, and he tells us that he has been requested for ten years to publish his exercises in decimall Arithmetike. After enlarging upon the advantages which attend the knowledge of this arithmetic to landlords and tenants, merchants and tradesmen, surveyors, gaugers, farmers, etc., and all men's affairs, whether by sea or land, he adds, "if God spare my life, I will spend some time in most cities of this land for my countries good to teach this art." This author was one of the earliest users of decimal fractions.

In the year 1619 there appeared, at Frankfort, a work on decimal arithmetic by Johann Hartman Beyern, in which the author states that he first thought upon the subject in the year 1597, but that he was prevented from pursuing it for many years by the little leisure afforded him from his professional pursuits. He makes no mention of Stevinus, but assumes throughout the invention as his own. The decimal places are indicated by the superscription of the Roman numerals, though the exponent corresponding to every digit in the decimal places is not always put down. Thus, 34.1426 is written $34^{\circ}.1^{I}\,4^{II}\,2^{III}\,6^{IV}$, or $34^{\circ}.14^{II}\,26^{IV}$, or $34^{\circ}.1426^{IV}$. The author must have been acquainted with the *Rabdologia* of Napier, as one chapter of his work is devoted to the explanation of the construction and use of these rods, which enjoyed a most extraordinary popularity at that period; and he could not, consequently, have been ignorant of Napier's notation or of the work of Stevinus; and we may therefore doubt the truth of his pretensions to being the originator of the system of decimals.

Albert Girard published an edition of the works of Stevinus in 1625, and in the solution of the equation $x^3=3x-1$ by a table of sines, of which method he was the author, we find the three roots as in the margin. On another occasion, he denotes the separation of the integers and decimals by a vertical line. He does not always adhere to this simple notation, as we afterwards find the square root of $4\frac{1}{3}$ expressed by 20816(4); and on another occasion we find similar vestiges of the original notation of Stevinus.

$$\left.\begin{array}{r}1,532\\347\\-1,879\end{array}\right\}$$

Oughtred is said to have contributed much to the propagation and general introduction of decimal arithmetic. In the first chapter of his *Clavis*, published in 1631, we find an explanation of decimal notation. The integers he separates from the *decimal*, or *parts*, by a mark, ∟, which he calls the *separatrix*, as in the examples, 0|56, 48|5, for .56 and 48.5; and in giving examples of the common operations of arithmetic he unites them under common rules. His view of the theory of decimals was generally adopted, and in some cases his notation also, by English writers on arithmetic for more than thirty years after this period.

In "Webster's tables for simple interest," etc., 1634, decimals seem to be treated as a thing generally known, though no decimal point is used. During the same year, 1634, Peter Herigone, of Paris, published a work in which he introduces the decimal fraction of Stevinus, having a chapter "des nombres de la dixme." The mark of the decimal is made by marking the place where the last figure comes. Thus when 137 livres 16 sous is to be taken 23 years 7 months, the product of $1378'$ and $23583'''$ is found to be $32497374''''$, or 3249 liv. 14 sous, 8 deniers. In 1633, John Johnson (Survaighor) published a work, the second part of which is called "Decimall Arithmatick wherby all fractionall operations are wrought, in whole numbers," etc. In his decimal fractions Johnson has the rudest form of notation; for he generally writes the places of decimals over the figures; thus, 146.03817 would be $146|\overset{1.}{0}\,\overset{2.}{3}\,\overset{3.}{8}\,\overset{4.}{1}\,\overset{5.}{7}$. In 1640, the "Arithmetica Practica" of Adrian Metius contains sexagesimal fractions, but not decimal ones; and a work by Joh. Henr. Alsted, in 1641, containing a slight treatise on arithmetic and algebra, says nothing about decimal fractions.

About this time the subject of decimals must have been pretty generally understood; for in "Moore's Arithmetick," 1650, the subject of decimals is quite thoroughly presented, and the contracted methods of multiplication and division are given. Noah Bridges, in his "Arithmetick Natural and Decimal," has an appendix on decimals, though the author expresses his disapproval of the use which some would make of decimals, averring that the rule of practice is more convenient in many cases. John Wallis, 1657, uses the old decimal notation $12|\underline{345}$, but he afterwards adopts the usual point in his algebra; and subsequently decimals seem to have been no longer regarded as a novelty, but took their place along with the other accepted subjects and methods of arithmetic.

It may be supposed that the publication of the tables of logarithms was necessarily connected with the knowledge and use of decimal arithmetic; but this, Dr. Peacock thinks, is not so. The theory of *absolute* indices, in its general form at least, was at that time unknown; and logarithms were not considered as the indices of the *base*, but as a measure of ratios merely. Under this view of their theory, it was a matter of indifference whether we assumed the measure of the ratio of 10 to 1 to be one, ten, a hundred, ten millions, or ten billions, the number assumed by Briggs in his system of logarithms. Thus, whether the logarithms are expressed by decimals or integers, they will have the same *characteristics*, and their use in calculation is exactly the same. It is under the integral forms that the logarithms are given in the earlier tables, such as those of Napier, Briggs, Kepler, etc.

This statement will sufficiently explain the reason why no notice is taken of decimals in the elaborate explanations which are given of the theory and construction of logarithms by Napier, Briggs, and Kepler; and indeed we find no mention of them in any English author between 1619 and 1631. In that year the *Logarithmical Arithmetike* was published by Gellibrand, a friend of Briggs who died the year before, with a much more detailed and popular explanation of the doctrine of logarithms than was to be found in Briggs's *Arithmetica Logarithmica*. It is there stated that the logarithms of 19695, 1969 $\frac{5}{10}$, 19$\frac{695}{1000}$ are respectively 4,29435 etc., 3,29435 etc., 1,29435 etc., differing merely in their characteristic; and $\frac{5}{10}$, $\frac{695}{1000}$, are called decimal fractions. Rules are also given for the reduction of vulgar fractions to decimals, by a simple proportion; and, lastly, a table for the reduction of shillings, pence, and farthings to decimals of a pound sterling.

The Decimal Point.—The final and greatest improvement in the system of decimal arithmetic, by which the notations of decimals and integers are assimilated, was the introduction of the decimal point, and much labor has been spent to ascertain its author. According to Dr. Peacock, the decimal point was introduced by Napier, the illustrious inventor of logarithms. In writing decimals Napier seems to have generally employed the method of Stevinus, which was to indicate the decimal places by primes, seconds, etc.; but there are at least two instances in which he used a character as a decimal separatrix. The first is an example of division in which he writes 1993,273, using a *comma*, and then presents his answer in the form 1993 2′ 7″ 3‴. The other instance occurs in a problem in multiplication, in which he draws a *line* down through the places of the partial products that would be occupied by the decimal point; but in the sum he uses the exponents of Stevinus, which thus combines both methods, and stands 1994 | $9'$ $1''$ $6'''$ $0''''$.

The problems in which these occur are found in the *Rabdologia*, published in 1617, in which he mentions the invention of Stevinus in terms of highest praise, and explains his notation without noticing his own simplification of it. The use of the comma, above referred to, is presented in the accompanying solution, in which it is required to divide 861094 by 432. I present but a part of the process of division.

```
        etc.
      118000
     141
    402
   429
   861094,000(1993,273
   432
   8888
    8888
     1296
     ----
     etc.
```

The quotient is 1993,273, or 1993,$2'$ $7''$ $3'''$

The use of the vertical *line* is found in an example of abbreviated multiplication which occurs in the solution of the following problem: "If 31416 be the approximate value of the circumference of a circle whose diameter is 10,000, what is the numerical value of the circumference of a circle whose diameter is 635?" This solution is said to be the first example found of this abbreviated multiplication; the use of it, however, became very popular in a short time afterward, being especially useful in the multiplication of the large numbers which were made use of in the construction of the tables of sines, etc.

This seems like a very near approach to the decimal point, if it is not indeed the introduction of it; but De Morgan maintains that Napier only used his comma or line as a *rest in the process*, and not as "a final and permanent indication, as well as a way of pointing out where the integers end and the fractions begin." It must be admitted that the use of the separatrix was merely incidental, and not the practice of Napier; but he seems to be the first to use a mark for this purpose, even incidentally, and there can be no doubt that even this incidental use had very great influence in leading to the general adoption of a decimal point.

De Morgan thinks that Richard Witt, who published a work four years before Napier, "made a nearer approach to the decimal point" than Napier; yet he says, "I can hardly admit him to have arrived at the notation of the decimal point." Witt, in a work published in 1613, presents some tables of compound interest, in which decimal fractions are used. The tables are constructed for ten millions of pounds, seven figures are cut off, and the reduction to shillings and pence with a *temporary* decimal separatrix, in the form of a vertical line, is introduced when wanted, as may be seen on page 446.

But though his tables are distinctly stated to contain only numerators, the denominator of which is always unity followed by ciphers, and though he had arrived at a complete and permanent command of the decimal separator, and though he always multiplies or divides by a power of 10 by changing the place of the decimal separator, which is a vertical line, yet De Morgan thinks he gave no "meaning to the quantity with its separator inserted." He thinks that if Witt had been "asked what his 123 | 456 was, he would have answered: It *gives* $123\frac{456}{1000}$, not it *is* $123\frac{456}{1000}$."

Briggs, the author of the common system of logarithms, was a disciple of Napier, and might have been expected to adopt Napier's method of writing decimals. We find, however, that in 1624, instead of using a decimal point he draws a line under the decimal terms, omitting the denominator; thus, $5\,\underline{9321}$. A work by Albert Girard, published in 1629 at Amsterdam, is remarkable as using the decimal point on a single occasion. Oughtred, in his *Clavis*, published in 1631, uses both the vertical and sub-horizontal separatrix, thus shutting up the numerator in a semi-rectangular outline, as $23\underline{|456}$ for 23.456. William Webster's work, published in 1634, treats of decimals as a thing generally known; but does not make use of the decimal point, using the partition line to separate integers and decimals. In 1657 John Wallis published a work in which the old notation, $12\,\underline{345}$, was used;

but he subsequently adopted the decimal point in his algebra. In 1643, the notation used in Johnson's arithmetic is £3|2 2 9 1 9 (with the figures 1 2 3 4 5 written over 2 2 9 1 9), and 3|2500, and 34,625, and sometimes 358|49411 fifths. Kavanagh says that the present notation was, for the first time, clearly set forth in some editions of Wingate's arithmetic, 1650. On the Continent the notation used was 12|345 or 12[345, even in works of the highest repute, up to the beginning of the 18th century.

The following summary presents some of the different methods of writing decimals which are found among the early writers on arithmetic, both in England and on the Continent:

34. 1′. 4″. 2‴. 6⁗	34 1426 (1426 underlined)
34. 1 . 4 . 2 . 6 (with (1) (2) (3) (4) written over 1, 4, 2, 6)	34 |1426 (1426 underlined)
34. 1 . 4 . 2 . 6	34‘1426 (1426 overlined)
34.1426⁗	34,1426
34.1426$^{(4)}$	

It is believed that Gunter, who was born in 1581, did more for the introduction of the decimal point than any one of his cotemporaries. He first adopted the notation of Briggs, but gradually dropped it and substituted the decimal point. In one of his works, De Morgan tells us, Briggs's notation appears without explanation, and 116 04 (04 underlined) is given as the third proportional to 100 and 108. On a subsequent page a dot is added to Briggs's notation in one instance; thus 100*l.* in 20 years at 8 per cent. becomes 466.095*l.* (095 underlined) At the bottom of the same page, Briggs's notation disappears thus: "It appeareth before, that 100*l.* due at the yeares end is worth but 92 592 (592 underlined) in ready money: If it be due at the end of two yeares, the present worth is 85*l.*733; then adding these two together, wee have 178*l.*326 for the present worth of 100 pound annuity for 2 yeares, and so forward." After this change, thus made without warning in the middle of a sentence, Briggs's notation does not again occur in the part of the work which relates to numbers. In a previous work on the sector, etc., the simple point is always *used;*

but in explanation the fraction is not thus written, but described as parts. Thus, 32.81 feet used in the operation is, in the description of the question or answer, 32 feet 81 parts.

It was some time after this, however, before the decimal point was fully recognized in all its uses, even in England; and on the Continent its introduction was even more tardy. As long as Oughtred was widely used, which was until the end of the seventeenth century, there must have been a large school of those who were trained to the notation 23|456. The complete and final victory of the decimal point must be referred to the first quarter of the eighteenth century. For a more detailed discussion of the subject, see the works of De Morgan and Peacock, from which what I have given is derived.

It may seem surprising that the decimal fraction should have been introduced so late in the history of the science; this delay, however, admits of explanation. The earliest division of the unit was into halves, thirds, etc.; the decimal division would be reached later, and the fraction based upon it would be of no special value, and could not be raised to the importance of a special class, until after the Arabic system of notation was adopted. Even then its introduction was necessarily tardy. Simple as they now appear to us, the invention of decimal fractions was too great an effort for one mind, or even one age. The idea of them and their use dawned gradually upon the mind; and one mathematician taking up what another had timidly begun, added an idea or two, until the subject was at length fully conceived and developed.

The advantages of the decimal notation of fractions are so obvious that they hardly need to be specified. Many of the operations upon fractions are thereby greatly simplified, and others are entirely avoided. The fundamental operations of addition, subtraction, multiplication and division, are the same as in integers, and the cases of reduction to lower terms, common denominator, etc., do not occur at all. The advantages would have been still greater if the basis of the numeral scale had been *twelve* instead of *ten*, as appears from a previous discussion.

CHAPTER II.

THE TREATMENT OF DECIMALS.

A DECIMAL FRACTION is a number of the decimal divisions of a unit; or it is a number of tenths, hundredths, etc. Some authors define it as a fraction whose denominator is ten or some power of ten; and others as a fraction whose denominator is one followed by one or more ciphers. Both of these definitions are correct, but seem less satisfactory than the one first presented. They are objectionable on account of not expressing the kind of fractional unit, but rather indicating its nature by describing the denominator of the fraction.

A Decimal Fraction may be expressed in two ways—in the form of a common fraction, or by means of the decimal scale. When expressed by the scale it is distinguished from the general meaning of the term *decimal fraction* by calling it a *Decimal.* A *Decimal* may thus be defined as a *decimal fraction* expressed by the decimal method of notation. Thus $\frac{5}{10}$, $\frac{45}{100}$, etc., are *decimal fractions,* but not *decimals;* while .5, .45, etc., are both *decimal fractions* and *decimals.* This distinction is convenient in practice, and is believed to be strictly logical. It has not been generally adopted, but there seems to be a growing tendency towards such a distinction. In popular language, however, we use the term "decimal fraction" as equivalent to a decimal.

Notation.—The decimal fraction, as expressed by the decimal scale, has no denominator written, the denominator being indicated by a point before the numerator. This notation, as already seen, arises from that of integers, and is merely an extension of it. Beginning at units' place, by a beautiful generalization, numbers are regarded as increasing toward the left and decreasing toward the right, in a ten-fold ratio, the result of which is a decimal division of the unit, corresponding to each decimal multiple of it.

In order to distinguish between the integral and fractional expression and locate each term properly, a point or separatrix is used. Various marks have been employed for this purpose, at different times, but the period is now generally adopted. The origin of this use of the decimal separatrix is discussed in the previous chapter. Sir Isaac Newton held that the point should be placed near the top of the figures, thus, 3·56, to prevent it from being confounded with the period used as a mark of punctuation.

Cases.—The cases in decimals, it is evident, must be nearly the same as in whole numbers. The relation of common fractions to decimals would, it is natural to suppose, give rise to one or more new processes. A new method of notation having been agreed upon for a special class of common fractions, the inquiry naturally arises,—Can other common fractions be expressed as decimals, and how? We thus begin to pass from common fractions to decimals; and, reversing this process, pass back from decimals to common fractions. This gives rise to a process known as the *Reduction of Fractions,* embracing the two cases of reducing common fractions to decimals, and its converse, decimals to common fractions. The reduction of common fractions to decimals gives rise to a particular kind of decimals called *circulates,* which require an independent treatment. The other cases of decimals are the same as in whole numbers.

Method of Treatment.—The method of treating decimals is quite similar to that of whole numbers. Indeed, they so closely resemble integers that many authors have been of the opinion that they should be presented with them. It is claimed that there is but one principle in the expression of integers and decimals, and that the processes and reasoning are the same, whether the scale is ascending or descending. It is therefore concluded that the notation of decimals should be presented with that of integers, and that the fundamental processes of addition, subtraction, etc., should be applied to them both in the same connection.

There are, however, valid objections to this seemingly plausible inference It will be admitted that the mechanical operations are the same; but the reasoning processes, in at least two of the fundamental operations, are not identical. The fixing of the decimal point in multiplication and division, would be entirely too difficult to be presented along with the fundamental operations of integers. Besides, it would be illogical to separate one class of fractions from the general subject of fractions; and moreover, one process, namely the reduction of decimals, could not be considered until after common fractions had been discussed. These considerations have been sufficient to prevent authors of arithmetic from uniting the treatment of decimals with that of integers, and will, I doubt not, continue to separate them.

Numeration.—In the treatment of decimals, the first thing to be considered is the method of reading and writing them, or their Numeration and Notation. These processes present several points worthy of notice, points which seem to have escaped the attention of the writers on arithmetic. Having introduced the subject of decimals by explaining that the first place to the right of units is tenths, the second place hundredths, etc., it immediately follows that .45 is read "4 tenths and 5 hundredths," but it does not immediately follow, as many arithmeticians are in the habit of assuming, that it is read "45 hundredths." If, however, it is first explained that $\frac{4}{10}$ is written .4, and $\frac{45}{100}$, .45, then it does not immediately follow that .45 is read "4 tenths and 5 hundredths." The usual method of presenting decimals is to explain that the first place to the right of the decimal point is *tenths*, the second place *hundredths*, etc.; it should then be shown that the decimal can be otherwise read. Thus, suppose we have the decimal .45: this expresses primarily 4 *tenths* and 5 *hundredths;* and since 4 *tenths* equals 40 *hundredths*, and 40 *hundredths* and 5 *hundredths* are 45 *hundredths*, the expression 45 may also be read 45 *hundredths*. This must be explained if we desire to preserve the chain of logical thought in our treatment.

From this it is seen that in practice there are two methods of reading decimals, which may be expressed as follows:

1. *Begin at the decimal point and read in succession the value of each term belonging to the decimal*, or

2. *Read the decimal as a whole number, and annex the name of the right-hand decimal place.*

It will be noticed that in reading a large decimal we should numerate *from* the decimal point to derive the denominator, and *toward* the decimal point to determine the numerator.

Notation.—The writing of decimals, when conceived or read to us, presents several points of interest. When the decimal is conceived analytically, that is, as so many tenths, hundredths, etc., we write it by the following rule:

1. *Fix the decimal point and write each term in its proper decimal place.*

If the decimal is conceived synthetically, that is, as a number of ten-thousandths, or a number of millionths, etc., we write it by the following rule:

2. *Write the numerator as an integer, and then place the decimal point so that the right-hand term shall express the denomination of the decimal.*

In writing a decimal in which the numerator does not occupy the required number of decimal places, it is not readily seen where to place the decimal point, and how many ciphers to prefix. The best practical rule in this case is the following.

3. *Write the numerator as an integer, and then begin at the right and numerate backward, filling vacant places with ciphers, until we reach the required denomination, and to the expression thus obtained, prefix the decimal point.*

Thus, to write 475 millionths, we first write 475; then beginning at the 5, we numerate toward the left, saying *tenths, hundredths, thousandths, ten-thousandths* (writing a cipher), *hundred-thousandths* (writing a cipher), *millionths* (writing a cipher), and then place the decimal point.

Several other methods have been suggested for writing decimals, among which is the following, by Prof. Henkle. It is seen that the *tens* of any number of *tenths*, the *hundreds* of any number of *hundredths*, the *thousands* of any number of *thousandths*, etc., each fall in the *order of units* when the decimal is expressed. Thus 56 *tenths*, is 5.6, the 5 *tens* falling in *units'* place; 2345 *hundredths* is 23.45, the 3 *hundreds* falling in *units'* place, etc. Hence the rule,

1. *Begin at the left and write the term corresponding to the denominator of the decimal in the place of units.*

Reduction.—The methods of treating the two cases of reduction are very simple. In reducing a common fraction to a decimal fraction, we reduce the different terms of the numerator to tenths, hundredths, etc., and divide by the denominator. In reducing a decimal to a common fraction, we express the decimal in the form of a common fraction, and then reduce it to its lowest terms.

Fundamental Operations.—Addition and subtraction are treated exactly as in integers, the same rules applying to both. The mechanical processes of multiplication and division are the same as in whole numbers; the only difference being the placing of the point in the product and quotient. There are two methods of explaining the location of the decimal point in multiplication and division, based upon the different conceptions of the origin of the decimal. One locates the point by the principles of common fractions; the other derives the method from the pure decimal conception. The latter is the simpler and more practical method. These two methods are explained in my works on written arithmetic, and need not be presented here.

CHAPTER III.

NATURE OF CIRCULATES.

THE adoption of the method of expressing fractions by the decimal scale opened up a new avenue of thought in the science of numbers. Decimals were treated without writing the denominator, and common fractions were frequently thrown into the decimal form and operated upon by means of the rules for whole numbers. The process of changing common fractions into the decimal scale led to the discovery of an interesting class of decimals called *Circulating Decimals.* These new forms soon attracted the attention and called forth the ingenuity of mathematicians; and, when investigated, were found to possess some remarkable and interesting properties.

Origin.—Circulating Decimals have their origin in the reduction of common fractions to decimals. In making this reduction, we annex ciphers to the numerator, and divide by the denominator. This division sometimes terminates with an exact quotient, and sometimes would continue on without ending. When it does terminate, the common fraction can be exactly expressed in a decimal; when it does not terminate, if the division be carried sufficiently far, a figure or set of figures will begin to repeat in the same order. Such a decimal is called a *circulating decimal,* or simply a *Circulate.*

It is thus seen that Circulates have their origin, not in the nature of number itself, but in the method of notation adopted to express numbers. They are an outgrowth of the Arabic system of notation and the decimal scale upon which it is based. If the scale of this system were duodecimal instead of decimal, the subject of Circulates would be greatly modified. Thus $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{9}$, etc., which now give circulates, would then give finite decimals; while $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{10}$, etc., would give circulating decimals.

Notation.—The part of the circulate which repeats is called a *Repetend* A Repetend is indicated by placing one or two periods or dots over it. A repetend of one figure is expressed by placing a point over the figure which repeats; thus $.\dot{3}$ expresses .333, etc. A repetend of more than one figure is expressed by placing a period over the first and the last figure; thus, $6.\dot{3}4\dot{5}$ expresses 6.345345, etc. Sometimes the first part of a decimal does not repeat, while the latter part does repeat. Such a decimal is called a *mixed circulate.* The part which repeats is called the *repeating part;* the part which does not repeat is called the *non-repeating* or *finite part* of the circulate. Thus $4.5\dot{3}\dot{6}$ is a mixed circulate in which 5 is the finite, or non-repeating part, and 36 the repeating part.

In an expression consisting of a whole number and a circulate, if the whole number contains terms similar to those of the repetend, the repetend may be indicated by placing one of the dots over a term in the whole number. Thus, suppose we have the circulate 54.234234, etc.; this is usually expressed thus, $54.\dot{2}3\dot{4}$; but, since the term just before the decimal point is the same as the last term of the repetend, it may also be expressed thus, $5\dot{4}.2\dot{3}$. This indicates that 423 repeats; and. expanding the expression, we have 54.23423 etc., which, expressed in the ordinary way, becomes $54.\dot{2}3\dot{4}$. In the same way, $\dot{6}.0\dot{4}$ denotes $6.\dot{0}4\dot{6}$; $\dot{2}0.1\dot{2}$ denotes $20.\dot{1}22\dot{0}$.

The reading of a repetend is a matter which often puzzles young teachers. Thus, in the case of the repetend $.\dot{3}$, since the denominator is 9, we cannot say "the decimal 3 tenths;" neither will it answer to say "the decimal 3 ninths;" how, then, shall it be read? The true reading is "the circulate 3 tenths." Calling it a *circulate* distinguishes it from the decimal fraction 3 tenths, and also indicates that it is equal to 3 ninths.

Again, how shall we read $.4\dot{3}\dot{6}$? It is not sufficiently explicit to say "the mixed circulate 436 thousandths," or "the mixed circulate 4 tenths and 36 thousandths," since neither of these expresses the idea exactly. The correct reading is, "the mixed circulate 436 thousandths, whose non-repeating part is 3 tenths and repeating part 36 thousandths." There may be other readings equally correct; the one suggested is given to lead teachers to avoid the adoption of those which are erroneous.

Definitions.—A *Circulate* is a decimal in which one or more figures repeat in the same order. A *Repetend* is the term or series of terms which repeat. This distinction between a *circulate* and a *repetend* should be carefully noted, as it is not always clearly understood. Circulates are *Pure* and *Mixed;* Repetends are *Perfect* and *Imperfect*, *Similar* and *Dissimilar*, and *Complementary*. A *Perfect Repetend* is one which contains as many decimal places, less one, as there are units in the denominator of the equivalent common fraction. Thus, $\frac{1}{7}=.\dot{1}4285\dot{7}$, and $\frac{1}{17}=.058823529411764\dot{7}$ are each perfect repetends.

Similar Repetends are those which *begin* and *end* respectively at the same decimal places; as $.4\dot{2}\dot{7}$ and $.5\dot{3}\dot{6}$. *Dissimilar Repetends* are those which begin or end at different decimal places. Especial attention is called to this definition of similar repetends, as it is a departure from the view usually taken. Repetends which *begin* at the same place have usually been regarded as *similar;* and those which *end* at the same place, *conterminous*. It is thought, however, to be much more precise to regard repetends beginning and ending respectively at the same places as similar. Repetends are surely not quite similar if they end at different places; to be similar they should both begin and end at the same place. This view makes it necessary to employ some other term to indicate a similarity of beginning. There being no word thus used, the term *coöriginous*, expressing a coörigin, is suggested. Its appropriateness may be seen by comparing it with *conterminous*, denoting a cotermination, which has already been adopted to denote a similarity of endings.

Cases.—Since circulates have their origin in the reduction of common fractions to decimals, it follows that the first case in the treatment of circulates is Reduction. The Reduction of Circulates embraces three distinct cases: 1. The reduction of common fractions to circulates; 2. The reduction of circulates to common fractions; 3. The reduction of dissimilar repetends to similar repetends. We have also Addition, Subtraction, Multiplication, and Division of Circulates. I have also recently introduced in my Higher Arithmetic the Greatest Common Divisor and Least Common Multiple of Circulates, subjects not heretofore treated in any arithmetical work. The comparison of circulates with common fractions gives rise to a number of interesting truths, which will be presented under the head of *Principles of Circulates.*

Method of Treatment.—The method of reducing common fractions to circulates is the same as that of reducing them to ordinary decimals. An abbreviation, based upon a principle of repetends, is sometimes employed. The method of reducing circulates to common fractions differs considerably from the method of reducing decimals to common fractions. In the finite decimal, the denominator understood is 1 with as many ciphers annexed as there are places in the decimal; in the circulate the denominator of the repetend is as many 9's as there are places in the repetend. There are three methods of explaining this reduction, as will be shown in the treatment.

Circulates can be added, subtracted, multiplied, and divided, by first reducing them to common fractions; or they may be expanded sufficiently far so that the repeating figures may appear in the result. Both of these methods are objectionable on account of their length, and are therefore not usually employed. In the addition and subtraction of circulates, it i better to reduce them to similar repetends and then perform the operation. In the multiplication and division of circulates, a slight modification of this method is employed.

CHAPTER IV.

TREATMENT OF CIRCULATES.

THE Treatment of Circulates embraces the operations of Reduction, Addition, Subtraction, Multiplication, Division, Greatest Common Divisor, Least Common Multiple, etc., and the Principles of Circulates. Attention will be called to the treatment of several of these subjects, and a distinct chapter will be devoted to the Principles of Circulates.

Reduction of Circulates.—The Reduction of Circulates is conveniently treated under four cases:

1. To reduce common fractions to circulates.
2. To reduce a pure circulate to a common fraction.
3. To reduce a mixed circulate to a common fraction.
4. To reduce dissimilar repetends to similar ones.

1. *To reduce common fractions to circulates.*—The general method of reducing common fractions to circulates is to annex ciphers to the numerator of the common fraction, and divide by the denominator, continuing the division until the figures of the circulate begin to repeat. Thus, to reduce $\frac{5}{12}$ to a circulate, we annex ciphers to the numerator 5, divide by the denominator 12, indicate the repeating figure by placing a period over it; and we have the circulate $.41\dot{6}$.

When the circulate consists of many figures, the process of reduction may be abbreviated by employing some of the principles of repetends. Thus, suppose it be required to reduce $\frac{1}{29}$ to a repetend. By actual division to five places, we find

$$\tfrac{1}{29}=0.03448\tfrac{8}{29}.$$

Now $\frac{8}{29}$ is 8 times $\frac{1}{29}$, hence multiplying this by 8 we have $\frac{8}{29}=0.27586\frac{6}{29}$. Substituting this value of $\frac{8}{29}$ in the expression for the value of $\frac{1}{29}$, and we have

$$\tfrac{1}{29}=0.0344827586\tfrac{6}{29}.$$

This, multiplied by 6, gives $\frac{6}{29}=0.2068965517\frac{7}{29}$; which, substituted in the second expression for $\frac{1}{29}$, gives

$$\tfrac{1}{29}=0.03448275862068965517\tfrac{7}{29}.$$

Multiplying by 7, we get $\frac{7}{29}=0.24137931034482758620\frac{18}{29}$; which, substituted in the third expression for $\frac{1}{29}$, gives

$$\tfrac{1}{29}=0.0344827586206896551724137931034482758620\tfrac{18}{29}.$$

As the terms have begun to repeat, it is unnecessary to continue the process any further. It will be seen, on examination, that the repetend consists of 28 figures, or one less than the denominator of $\frac{1}{29}$, and therefore is a perfect repetend.

2. *To reduce a pure circulate to a common fraction.*—There are three distinct methods of explaining this case, as has already been stated. In order to illustrate these methods, we will solve the problem, Reduce $.\dot{4}\dot{5}$ to a common fraction.

In the first method, having proved by actual division that $.\dot{1}=\frac{1}{9}$, $.\dot{0}\dot{1}=\frac{1}{99}$, $.\dot{0}0\dot{1}=\frac{1}{999}$, etc., we derive the denominator of any circulate from its relation to these given circulates. To illustrate, reduce the circulate $.\dot{4}\dot{5}$ to a common fraction. The method is as follows: Since $.\dot{0}\dot{1}=\frac{1}{99}$, as shown by actual division, $.\dot{4}\dot{5}$, which is 45 times $.\dot{0}\dot{1}$, equals 45 times $\frac{1}{99}$, or $\frac{45}{99}$, which, reduced to its lowest terms, equals $\frac{5}{11}$.

OPERATION.

$$.\dot{0}\dot{1}=\tfrac{1}{99}$$

$$.\dot{4}\dot{5}=\tfrac{45}{99}=\tfrac{5}{11}.$$

By the second method, we multiply the circulate by 1 with as many ciphers annexed as there are places in the repetend, which makes a whole number of the repeating part of the circulate. We then subtract the two circulates, and have a certain number of times the given circulate equal to the difference, from which the given circulate is readily found. We will illustrate by the solution of the same problem.

Let C represent the common fraction which equals the circulate; we will then have C=.4545 etc.; multiplying by 100 to make a whole number of the repeating part, we have 100 times the common fraction equal to 45.4545 etc.; subtracting once the common fraction from 100 times the common

OPERATION.

$$\begin{aligned} C&=.4545\text{ etc.}\\ 100C&=45.4545\text{ etc.}\\ \hline 99C&=45\\ C&=\tfrac{45}{99}=\tfrac{5}{11}. \end{aligned}$$

fraction, we have 99 times the common fraction equal to 45.4545 etc., minus .4545 etc., which equals 45; hence once the common fraction equals $\frac{45}{99}$, or $\frac{5}{11}$.

By the third method, the repetend is regarded as an infinite series, the ratio being a fraction whose numerator is 1, and denominator 1 with as many ciphers annexed as there are places in the repetend. The solution is as follows: The repetend $.\dot{4}\dot{5}$ may be regarded as an infinite series, $\frac{45}{100}+\frac{45}{10000}+$etc. The formula for the sum of an infinite series is $S=\frac{a}{1-r}$.

OPERATION.

$$.\dot{4}\dot{5}=\frac{45}{100}+\frac{45}{10000}+\text{etc.}$$
$$S=\frac{a}{1-r}=\frac{45}{100}\div\frac{99}{100}$$
$$=\frac{45}{99}=\frac{5}{11}.$$

Substituting the value of $a=\frac{45}{100}$, and $r=\frac{1}{100}$, we have $S=\frac{45}{100}\div\frac{99}{100}$, which equals $\frac{45}{99}$, or $\frac{5}{11}$.

3. *To reduce a mixed circulate to a common fraction.*—There are three distinct methods of reducing mixed circulates to common fractions, as in the preceding case. To illustrate these methods we will solve the problem, Reduce $.3\dot{1}\dot{8}$ to a common fraction. By the first method, we reason thus: The mixed circulate $.3\dot{1}\dot{8}$ equals $\frac{1}{10}$ of $3.\dot{1}\dot{8}$, which by the preceding case equals $\frac{1}{10}$ of $3\frac{18}{99}$, or $\frac{1}{10}$ of $3\frac{2}{11}$, which equals $\frac{35}{110}$, or $\frac{7}{22}$.

OPERATION.

$$.3\dot{1}\dot{8}=\frac{1}{10}\text{ of }3.\dot{1}\dot{8}$$
$$=\frac{3\frac{18}{99}}{10}=\frac{3\frac{2}{11}}{10}$$
$$=\frac{35}{110}=\frac{7}{22}.$$

By the second method, we reason as follows: Let C represent the common fraction, then we shall have C=.31818 etc.; multiplying by 10 to make a whole number of the non-repeating part, we have 10 times the fraction equals 3.1818 etc.; multiplying this by 100 to make a whole number of the repeating part, we have 1000 times the fraction equals 318.1818 etc.; subtracting 10 times the fraction from 1000 times the fraction, we have 990 times the fraction equals 315, from which we find the fraction equals $\frac{315}{990}$, or $\frac{7}{22}$.

OPERATION.

$$C=\quad .31818\text{ etc.}$$
$$10C=\quad 3.1818\text{ etc.}$$
$$1000C=318.1818\text{ etc.}$$
$$990C=315$$
$$C=\frac{315}{990}=\frac{7}{22}.$$

In the previous method we see that we subtract the finite part from the entire circulate, and divide by as many 9's as there are figures in the repetend, with as many ciphers annexed as there are decimal places before the repetend; hence, by generalizing this into a rule, we may perform the operation as in the margin. This is the method preferred in practice.

OPERATION.

$$.3\dot{1}\dot{8}$$
$$.318$$
$$3$$
$$\frac{315}{990}=\frac{7}{22}.$$

This case may also be solved by regarding the repetend as an infinite series, and finding its sum by geometrical progression, and then adding it to the finite part. The solution is presented in the margin, in which it is seen that we regard $\frac{18}{1000}$ as the first term of the series, and $\frac{1}{100}$ as the rate.

OPERATION.

$$.3\dot{1}\dot{8}=\frac{3}{10}+\frac{18}{1000}+\frac{18}{100000}+\text{etc.}$$
$$S=\frac{18}{1000}\div\frac{99}{100}=\frac{18}{990}$$
$$\frac{3}{10}+\frac{18}{990}=\frac{315}{990}=\frac{7}{22}.$$

4. *To reduce dissimilar repetends to similar ones.* To solve this case it is necessary to understand the following principles:

1. Any terminate decimal may be considered interminate, its repetend being ciphers; thus, $.45=.45\dot{0}$, or $.45\dot{0}0\dot{0}$, etc.

2. A simple repetend may be made compound by repeating the repeating figure; thus, $.\dot{3}=.\dot{3}\dot{3}=.\dot{3}33\dot{3}$, etc.

3. A compound repetend may be enlarged by moving the right-hand dot towards the right over an exact number of periods; thus, $.2\dot{4}\dot{5}=.2\dot{4}54\dot{5}$, etc.

4. Both dots of a repetend may be moved the same number of places to the right; thus, $.5\dot{3}7\dot{8}=.53\dot{7}8\dot{3}$ or $.537\dot{8}3\dot{7}$, etc., for each expression developed will give the same result.

5. Dissimilar repetends may be made coöriginous by moving both dots of the repetend to the right until they all begin at the same place.

6. Dissimilar repetends may be made conterminous by moving the right-hand dots of each repetend over an exact number of periods of each repetend until they end at the same place.

The method of treating this case may be illustrated by the following example: Make $.\dot{4}\dot{5}$, $.4\dot{3}6\dot{2}$, and $.81\dot{3}69\dot{4}$ similar. The solution is as follows: To make these repetends similar, they must be made to begin and end at the same place. To do this, we first move the left-hand dots so that they begin at the same place, and then move the right-hand dots over an exact number of periods, so that they will end at the same place. Now the number of places in the periods are respectively 2, 3, and 4; hence the number of places in the new periods must be a common multiple of 2, 3, and 4, which is 12; we therefore move the right-hand dot so that each repetend shall contain 12 places.

OPERATION.

$$.\dot{4}\dot{5} = .45\dot{4}545454545\dot{4}5$$

$$.4\dot{3}6\dot{2} = .43\dot{6}2362362362\dot{3}$$

$$.81\dot{3}69\dot{4} = .81\dot{3}6943694369\dot{4}$$

Divisor and Multiple.—The Greatest Common Divisor of two or more decimals is the greatest decimal that will exactly divide them. Such a divisor can be found by reducing the decimals to common fractions, and applying the method for common fractions; but it can also be found by keeping them in the decimal form; and the latter method is generally less tedious and more direct. To illustrate the method, let us find the greatest common divisor of $.3\dot{7}\dot{5}$ and $.4\dot{2}\dot{3}$. We make the two circulates similar, and subtract the finite part, which reduces them to fractions having a common denominator. We then find the greatest common divisor of their numerators, 1638, which is the numerator of the greatest common divisor, the denominator being of the same denomination as the similar decimals; hence the greatest common divisor is $\frac{1638}{9999990}$, or $.\dot{0}00163\dot{8}$.

OPERATION.

$.3\dot{7}57575\dot{5}$	$.4\dot{2}34234\dot{4}$	
3	4	
3757572	4234230	1
	3757572	
3813264	476658	8
55692	501228	9
49140	24570	2
6552	26208	4
6552	1638	4

$$\frac{1638}{9999990} = .\dot{0}00163\dot{8}, \text{ G. C. D}$$

The method, it is seen, consists in reducing the decimals to a common denominator, finding the greatest common divisor of their numerators, writing this over the common denominator, and reducing the resulting fraction to a decimal.

The Least Common Multiple of two or more decimals is the least number that will exactly contain each of them. Such a multiple can be found by reducing the decimals to common fractions and applying the method for common fractions; but it can also be found by keeping them in their decimal form; and the latter method is preferred, as being generally more direct and less laborious.

To illustrate the method, let us find the least common multiple of $.3\dot{2}\dot{7}$, $\dot{1}.01\dot{1}$ and $.0\dot{7}\dot{5}$. We reduce the circulates to fractions having a common denominator, as in the previous case. The least common multiple of these numerators is 275699700, which is the numerator of the least common multiple, the denominator being the common denominator of the fractions. Reducing $\frac{275699700}{99990}$, the least common multiple, to whole numbers and decimals, we have $275\dot{7}.\dot{2}$, the least common multiple.

OPERATION.

	$.3\dot{2}72\dot{7}$	$1.0\dot{1}11\dot{0}$	$.0\dot{7}57\dot{5}$
	3	10	0
3	32724	101100	07575
4	10908	33700	2525
25	2727	8425	2525
101	2727	337	101
	27	337	1

$$3\times4\times25\times101\times27\times337=275699700$$

$$\frac{275699700}{99990}=2757.\dot{2}72\dot{7}, \text{ L. C. M.}$$

$$=275\dot{7}.\dot{2}$$

It will be seen that the method consists in reducing the decimals to a common denominator, finding the least common multiple of their numerators, writing this over the common denominator, and reducing the resulting fraction to a decimal.

CHAPTER V.

PRINCIPLES OF CIRCULATES.

THE investigation of the relation of circulate forms to common fractions has led to the discovery of some very interesting and remarkable properties. These will be considered under the head of *Principles of Circulates*, and *Complementary Repetends.* The subject being rather briefly treated in the text-books, will be presented here somewhat in detail. A brief and simple explanation will be given in connection with each principle.

1. *A common fraction whose denominator contains no other prime factors than 2 or 5, can be reduced to à simple decimal.* For, since 2 and 5 are factors of 10, if we annex as many ciphers to the numerator as there are 2's or 5's in the denominator, the numerator will then be exactly divisible by the denominator.

2. *The number of places in the simple decimal to which a common fraction may be reduced, is equal to the greatest number of 2's or 5's in the denominator.* For, to make the numerator contain the denominator, we must annex a cipher for every 2 or 5 in the denominator, and the number of places in the quotient, which is the decimal, will equal the number of ciphers annexed.

3. *Every common fraction, in its lowest terms, whose denominator contains other prime factors than 2 or 5, will give an interminate-decimal.* For, since 2 and 5 are the only factors of 10, if the denominator contains other prime factors, the numerator with ciphers annexed will not exactly contain the denominator; hence the division will not terminate, and the result will be an interminate decimal.

4. *Every common fraction which does not give a simple decimal, gives a circulate.* For, in reducing a common fraction to a decimal, there cannot be more different remainders than there are units in the denominator; hence, if the division be continued, a remainder must occur which has already been used, and we shall thus have a series of remainders and dividends like those already used, therefore the terms of the quotient will be repeated.

5. *The number of figures in a repetend cannot exceed the number of units in the denominator of the common fraction which produces it, less* 1. For, in reducing a common fraction to a decimal, when the number of decimal places equals the number of units in the denominator, less 1, all the possible different remainders will have been used, and hence the dividends, and therefore the quotients which constitute the circulate, will begin to repeat. In many cases the remainders begin to repeat before we have as many as the denominator less 1.

6. *The number of places in a repetend, when the denominator of the common fraction producing it is a prime, is always equal to the number of units in the denominator, less* 1, *or to some factor of this number.* For, the repetend must end when it reaches the point where it has as many places less 1 as there are units in the denominator of the producing fraction; hence, if it ends before this, the number of places must be an exact part of the number of places in the denominator less 1, that it may terminate when it has as many places as the denominator less 1. This is not generally true when the denominator is composite, as in $\frac{1}{21}$, $\frac{1}{33}$, $\frac{1}{39}$, $\frac{1}{49}$, etc.

7. *A common fraction whose denominator contains 2's or 5's with other prime factors, will give a mixed circulate, and the number of places in the non-repeating part will equal the greatest number of 2's or 5's in the denominator.* Dividing first by the 2's and 5's, we shall have a decimal numerator containing as many places as the greatest number of 2's or 5's

(Prin. 2). If we now divide by the other factors, the dividends consisting of the terms of the decimal numerator will not give the same series of remainders as when we have a series of dividends with ciphers annexed; hence the circulate will begin directly after the last place of these decimal terms. To illustrate, take $\frac{1}{350}$, and factor the denominator, and we have

$$\frac{1}{350}=\frac{1}{2\times5\times5\times7};$$

dividing by the 2 and the 5's we have $\frac{.02}{7}$, in which it is evident the circulate must begin in the third decimal place, just as the circulate from $\frac{2}{7}$ begins in the first decimal place.

8. *When the reciprocal of a prime number gives a perfect repetend, the remainder which occurs at the close of the period is* 1. For, since the reduction of the fraction to a circulate commenced with a dividend of 1 with one or more ciphers annexed, that the quotients may repeat we must begin with the same dividend, and therefore the remainder at the close of the period must be 1.

9. *When the reciprocal of any prime number is reduced to a repetend, the remainder which occurs when the number of decimal places is one less than the prime, is* 1. For, since the number of decimal places in the period equals the denominator less 1, or is a factor of the denominator less 1, at the close of a period consisting of as many places as the denominator less 1, there will be an exact number of repeating periods, and therefore the remainder will be 1.

10. *A number consisting of as many 9's as there are units in any prime less* 1, *is divisible by that prime.* For, if we divide 1 with ciphers annexed by a prime, after a number of places 1 less than the prime, the remainder is 1; hence 1 with the same number of ciphers annexed, minus 1, would be exactly divisible by the prime; but this remainder will be a series of 9's, therefore such a series of 9's is divisible by the prime. Thus 999999 is divisible by 7.

11. *A number consisting of as many* 1's *as there are units*

in any prime (except 3), less 1, is divisible by that prime. For the prime is a divisor of a series of 9's (Prin. 10), which is equal to 9 times a series of 1's; and since 9 and the prime are relatively prime, and the prime is a divisor of 9 times a series of 1's, it must be a divisor also of a series of 1's. Thus 111111 is divisible by 7; also 1111111111 is divisible by 11.

12. *A number consisting of any digit used as many times as there are units in a prime (except 3), less 1, is divisible by that prime.* For, since such a series of 1's is divisible by the prime, any number of times such a series of 1's will be divisible by the prime. Hence 222222, 333333, 444444, etc., are each divisible by 7.

13. *The same perfect repetend will express the value of all proper fractions having the same prime denominator, by starting at different places.* Thus, $\frac{1}{7}=.14285714285$ etc. But $\frac{1}{7}=.1\frac{3}{7}$, hence the part that follows 1 in the repetend of $\frac{1}{7}$ is the repetend of $\frac{3}{7}$; that is, $\frac{3}{7}=.\dot{4}2857\dot{1}$. Again, $\frac{1}{7}=.14\frac{2}{7}$; hence the part that follows .14 in the repetend of $\frac{1}{7}$ is the repetend of $\frac{2}{7}$; that is, $\frac{2}{7}=.\dot{2}8571\dot{4}$. In a similar manner we find $\frac{6}{7}=.\dot{8}5714\dot{2}$, $\frac{4}{7}=.\dot{5}7142\dot{8}$; and the same thing is generally true.

14. *In reducing the reciprocal of a prime to a decimal, if we obtain a remainder 1 less than the prime, we have one-half of the repetend, and the remaining half can be found by subtracting the terms of the first half respectively from* 9. Take $\frac{1}{7}$, and let us suppose in decimating we have reached a remainder of 6; now what follows will be the repetend of $\frac{6}{7}$, and the repetend of $\frac{6}{7}$ added to the repetend of $\frac{1}{7}$ must equal 1, since $\frac{6}{7}+\frac{1}{7}=1$; hence the sum of these two repetends must equal .999999 etc., since .999999 etc. equals 1. Now in adding the terms of these two repetends together, that the sum may be a series of 9's, there must be just as many places before the point where 6 occurred as a remainder, as after; hence 6 occurred as a remainder when we were half through the series.

Again, since the sum of the terms of the latter and the former half of the repetend equals a series of 9's, each term of the first half of the repetend, subtracted from 9, will give the corresponding term of the latter half of the series.

All perfect repetends possess this property, and a large number of those which are not perfect. Repetends possessing this property are called *complementary repetends.* The last two properties are of great practical value in reducing common fractions to repetends.

15. *Any prime is an exact divisor of* 10 *raised to a power denoted by the number of terms in the repetend of the prime, less* 1; *or of* 10 *raised to a power denoted by any multiple of the number of terms, less* 1. For, by Prin. 6, the number of places in the repetend must equal the number of units in the prime, or some factor of that number; hence the dividend used in obtaining a period must be 10 raised to a power equal to the number of terms in the period; and since the remainder at the end of the period is 1, the prime will exactly divide 10 raised to a power equal to the number of terms in the period, less 1.

Both this and principle 6 depend on *Fermat's Theorem,* that "$P^{p-1}-1$ is divisible by p when p and P are prime to each other." For 10, the base of the decimal system, is prime to any prime number except 2 and 5; hence $10^{p-1}-1$ is always exactly divisible by p, when p is any prime except 2 and 5. It thus follows that in the division of 1 with ciphers annexed, the remainder is always 1 when the number of places in the quotient is equal to the number of units in the prime. From this we can readily derive the second part of principle 6, and also principle 15.

16. *Any prime is an exact divisor of a number when it will divide the sum of the numbers formed by taking groups of the number consisting of as many terms as there are figures in the repetend of the reciprocal of that prime.* We will show this for a prime whose reciprocal gives a repetend of three places. The number 47,685,672,856, may be put in the form $856+672\times10^3+685\times10^6+47\times10^9$, or $672\times(10^3-1)+685\times(10^6-1)+47\times(10^9-1)+856+672+685+47$; but these

different powers of 10, diminished by 1, are all divisible by any number whose reciprocal gives a number of three places, as 37; hence if the sum of the groups, 47+685+672+856, is divisible by 37, the entire number is also divisible by 37. The same may be illustrated with any other number, and the principle is therefore general. The principle admits, also, of a general demonstration.

From this general proposition we derive the following special principles embraced under it:

1. Since the reciprocals of 3 and 9 give a repetend of one place, they will divide a number when they divide the sum of the digits.

2. Since the reciprocals of 11, 33, and 99, give a repetend of two places, they will divide a number when they divide the sum of the numbers found by taking groups of two places.

3. Since the reciprocals of 27, 37, and 111, give repetends of three places, they will divide a number when they divide the sum of the numbers formed by taking groups of three places.

4. Since the reciprocal of 101 gives a repetend of four places, it will divide a number when it divides the sum of the numbers formed by taking groups of four places.

5. Since the reciprocals of 41 and 271 give repetends of five places, they will divide a number when they divide the sum of the numbers formed by taking groups of five places.

6. Since the reciprocals of 7, 13, 21, and 39 give repetends of six places, they will divide a number when they divide the sum of the numbers formed by taking groups of six places.

CHAPTER VI.

COMPLEMENTARY REPETENDS.

COMPLEMENTARY REPETENDS are those in which the terms of the first half of the period are respectively equal to 9 minus the corresponding terms of the second half of the period. Thus, in the repetend arising from $\frac{1}{7}$, which is $.\dot{1}4285\dot{7}$, the first term, 1, subtracted from 9, leaves the fourth term, 8; the second term 4, subtracted from 9, leaves the fifth term, 5, etc. Complementary Repetends include all perfect repetends, and many repetends that are not perfect. From the principles presented in the preceding chapter, the following curious properties of complementary repetends will be readily understood:

1. *If the last half of the terms of a perfect repetend be written in order under the first half and added to the terms in the first half, the sum will be a succession of 9's.* Thus, the fraction $\frac{1}{23}=.\dot{0}434782608695652173913\dot{}$; and this repetend, written and added as suggested, will give a series of 9's, as is seen in the margin.

OPERATION.

04347826086
95652173913
99999999999

2. *If the remainders obtained in reducing the common fraction to a repetend be written in the same way and added, each sum will be the denominator of the common fraction.* Thus, the remainders in reducing $\frac{1}{23}$ are

	10,	8,	11,	18,	19,	6,	14,	2,	20,	16,	22,
	13,	15,	12,	5,	4,	17,	9,	21,	3,	7,	1, which, added,
give	23,	23,	23,	23,	23,	23,	23,	23,	23,	23,	23.

3. *If we subtract the unit term of the denominator of the common fraction from* 10, *and multiply any term of the repe-*

tend by the remainder, the unit term of the product will be the unit term of the corresponding remainder. Thus, in $\frac{1}{23}$,

.0, 4, 3, 4, 7, 8, 2, etc., are the terms of the repetend.
10 − 3 = 7
0, 8, 1, 8, 9, 6, 4, etc., unit terms of products and remainders.

4. *A complementary repetend, by beginning at different points, will be the repetend of all proper fractions having the same denominator as the fraction which produced it.* Thus, $\frac{1}{23}=.\dot{0}43478208$ etc.; and $\frac{10}{23}=.\dot{4}3478$ etc., which begins with the second figure of the circulate of $\frac{1}{23}$. Again, $\frac{8}{23}=.\dot{3}4782$ etc., which begins with the third figure of the circulate equal to $\frac{1}{23}$ etc.

5. *The numerator of the fraction equal to any one of the several repetends beginning with the successive figures of a complementary repetend, is the remainder left when the preceding figure of the repetend was obtained.* Thus, in the circulate of $\frac{1}{23}$, when the first 4 of the circulate was obtained, 8 was the remainder, and 8 is the numerator of the fraction equal to the circulate $.\dot{3}4782$ etc.

The following are all the perfect repetends whose denominators are less than 100:

$\frac{1}{7}=.\dot{1}4285\dot{7}$

$\frac{1}{17}=.\dot{0}58823529411764\dot{7}$

$\frac{1}{19}=.\dot{0}5263157894736842\dot{1}$

$\frac{1}{23}=.\dot{0}43478260869565217391\dot{3}$

$\frac{1}{29}=.\dot{0}34482758620689655172413793\dot{1}$

$\frac{1}{47}=.\dot{0}21276595744680851063829787234042553191489361\dot{7}$

$\frac{1}{59}=\begin{cases}.\dot{0}16949152542372881355932203389830508474576271186\\ 440677966\dot{1}\end{cases}$

$\frac{1}{61}=\begin{cases}.\dot{0}16393442622950819672131147540983606557377049180\\ 32786885245\dot{9}\end{cases}$

$\frac{1}{97}=\begin{cases}.\dot{0}10309278350515463917525773195876288659793814432\\ 98969072164948453608247422680412371134020618556\dot{7}\end{cases}$

The following repetends are complementary, as may be seen by the test for complementary repetends, but are not perfect:

$\frac{1}{11}=.\dot{0}\dot{9}$

$\frac{1}{13}=.\dot{0}7692\dot{3}$

$\frac{1}{73}=.\dot{0}136986\dot{3}$

$\frac{1}{89}=.\dot{0}112359550561797752808$ {Half a Period.

$\frac{1}{101}=.\dot{0}09\dot{9}$

$\frac{1}{103}=.\dot{0}097087378640776 6$ {Half a Period.

The following common fractions give an even number of figures in a period, but the repetends are not complementary, as will be seen by applying the test for complementary repetends:

$\frac{1}{21}=.\dot{0}4761\dot{9}$

$\frac{1}{33}=.\dot{0}\dot{3}$

$\frac{1}{39}=.\dot{0}2564\dot{1}$

$\frac{1}{49}=.\dot{0}204081$ etc., to 42 places.

$\frac{1}{51}=.\dot{0}19607843137254\dot{9}$.

The following are not complementary repetends, but the number of places in each repetend is an exact part of the number of units less one in the denominator of the common fraction producing it:

$\frac{1}{31}=.\dot{0}3225806451612\dot{9}$

$\frac{1}{41}=.\dot{0}243\dot{9}$

$\frac{1}{43}=.\dot{0}2325581395348837209\dot{3}$

$\frac{1}{53}=.\dot{0}18867924528\dot{3}$

$\frac{1}{67}=.\dot{0}14925373134328358208955223880559\dot{7}$

$\frac{1}{71}=.\dot{0}14084507042253521126760563380281 6\dot{9}$

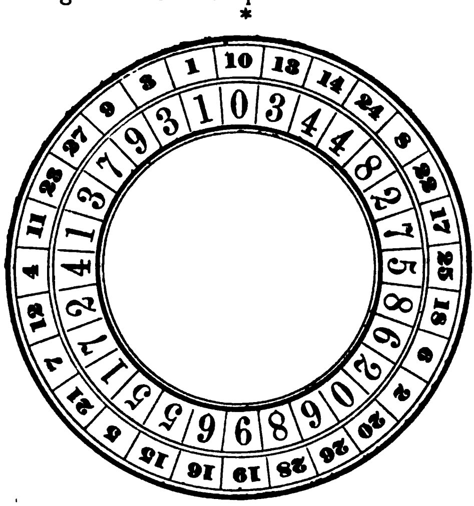
$\frac{1}{79}=.\dot{0}126582278 48\dot{1}$

$\frac{1}{83}=.\dot{0}12048192771084337349397590361445783132 5\dot{3}$

$\frac{1}{107}=\begin{cases}.\dot{0}093457943925233644859813084112149532710280373 8\\ 31775\dot{7}\end{cases}$

Professor Perkins has ingeniously illustrated some of the properties of repetends, by arranging the terms of a repetend and the corresponding remainders which arise in obtaining them, in concentric circles.

The following illustration and discussion I have taken from his work on Higher Arithmetic published in 1844:



In this figure, the inner circle of figures, commencing at the 0, directly under the asterisk, and counting towards the right hand, is the circulating period of $\frac{1}{29}$.

The outer circle of figures, commencing at the same place and counting in the same direction, are the successive remainders which will occur in the operation of decimating $\frac{1}{29}$.

In this circle of remainders, all the numbers from 1 to 28, inclusive, occur, but not in numerical order.

From what has been previously explained, we infer the following properties, which are common to all perfect repetends:

1. *The sum of any two diametrically opposite terms of the circle of decimals will be* 9.

2. *The sum of any two diametrically opposite terms in the circle of remainders will equal the denominator* 29.

3. *If we subtract the right-hand term of the denominator from* 10, *and multiply the remainder by any decimal term of*

the inner circle, the right-hand figure of the product will be the same as the right-hand figure of the corresponding remainder of the outer circle.

4. *Commencing the circle of decimals at any point and counting completely round, we shall have the perfect repetend of the common fraction whose denominator is the same as in the first case, but whose numerator is the remainder in the outer circle standing one place to the left.*

5. *If we divide the product of any two remainders by 29, what remains will be the remainder in the outer circle, corresponding with the place denoted by the sum of the places of the two numbers.*

From the fourth property it follows that this same circle of decimals expresses the decimal value of all proper common fractions whose denominators are 29.

A similar figure, formed from the perfect repetend of the common fraction $\frac{1}{28}$, possesses properties the same as those just explained. Similar circles may be formed for all perfect repetends.

If we arrange the complementary repetend arising from the fraction $\frac{1}{211}$ in the form of a circle, as was done for perfect repetends, it will be seen that a complementary repetend possesses all the properties ascribed to the perfect repetend on pages 479 and 480, except the fourth.

CHAPTER VII.

A NEW CIRCULATE FORM.

IN The Normal Written Arithmetic, I presented two or three curious circulate forms which have given rise to so much discussion among teachers, that it is thought well to call attention to them in this work. These forms are $.0\dot{\tfrac{1}{2}}$ and $.\dot{0}\tfrac{1}{2}0\dot{\tfrac{1}{3}}$, both of which are intended to represent pure circulates. Similar forms may also occur in mixed circulates, as $.0\tfrac{1}{2}0\dot{\tfrac{1}{3}}$ and $.0\tfrac{1}{2}\dot{0}\tfrac{1}{3}0\dot{\tfrac{1}{4}}$. Two questions have been raised with regard to these expressions: first, what do they mean? and second, are they legitimate arithmetical expressions? I propose to say a few words on their *meaning*, their *origin*, and their *value*.

Before discussing this circulate form, attention is called to the signification of the decimal expression $.2\tfrac{1}{2}$. In this expression, does the $\tfrac{1}{2}$ express *tenths* or *hundredths?* Is $\tfrac{1}{2}$ regarded as occupying one of the places in the decimal scale, or is it a part of the tenth? It has been held that the $\tfrac{1}{2}$ occupies hundredths place; but a very slight consideration is sufficient to lead one to see that the $\tfrac{1}{2}$ is one-half of a tenth, and not one-half of a hundredth. In integers and fractions, the fraction always denotes a part of a unit of the term to the right of which it stands. Thus, in $2\tfrac{1}{2}$, the 2 expresses units, and the $\tfrac{1}{2}$ is $\tfrac{1}{2}$ of a unit. Now, if we divide $2\tfrac{1}{2}$ by 10, we have $.2\tfrac{1}{2}$; in which the .2 expresses *tenths*, and by the principle, as before, the $\tfrac{1}{2}$ would be $\tfrac{1}{2}$ of a *tenth*. Conversely, if we take the expression $.2\tfrac{1}{2}$ and multiply it by 10, we have $.2\tfrac{1}{2}\times 10=2\tfrac{1}{2}$, and not 2 units and $\tfrac{1}{2}$ of a tenth. The same result will be reached if, regarding the $\tfrac{1}{2}$ as $\tfrac{1}{2}$ of a tenth, we reduce it to 5 hundredths, in which case the expression becomes .25; for multiplying this by 10, we have $.25\times10=2.5$, or $2\frac{1}{2}$. Now, if the above is true for $.2\frac{1}{2}$, it is equally true for $.1\frac{1}{2}$ or $.0\frac{1}{2}$; from which we see that in the expression $.0\frac{1}{2}$, the $\frac{1}{2}$ belongs to tenths place, and should not be regarded as occupying hundredths place.

These expressions may actually arise in an investigation. Suppose we are required to subtract \$12.62 from \$25.62$\frac{1}{2}$; the difference is evidently \$13.00$\frac{1}{2}$; also 3.4 subtracted from $6.4\frac{1}{2}$ equals $3.0\frac{1}{2}$. In the former case it is clear that the $\frac{1}{2}$ is a half of a hundredth; in the latter, a half of a tenth. It hardly seems necessary, after this, to say that $3.\frac{1}{2}$ does not express 3 and $\frac{1}{2}$ of a tenth, as some have supposed; and also that the expression is an illegitimate one, without any other meaning than $3\frac{1}{2}.00$. Let us now consider the forms mentioned above.

Origin.—These circulate forms originated from an extension of a mixed integer to a mixed decimal, and then a further extension of the decimal to a circulate. They may, however, be immediately derived from the reduction of a common fraction to a decimal. To illustrate, reduce $\frac{1}{18}$ to a decimal. Following the ordinary rule, we annex zeros to the numerator and divide by the denominator: 18 is contained in 10 tenths $\frac{1}{2}$ of a tenth time, with 1 tenth or 10 hundredths as a remainder; 18 is contained in 10 hundredths $\frac{1}{2}$ of a hundredth time, with 1 hundredth remaining, etc. Here we observe that the remainders repeat; hence the quotient figures will repeat, and $\frac{1}{18}$ may be regarded as equal to $.0\frac{1}{2}0\frac{1}{2}0\frac{1}{2}$ etc., or $.0\dot{\frac{1}{2}}$.

OPERATION.

$$
\begin{array}{r l}
18)1.00(.0\frac{1}{2}0\frac{1}{2}, \text{ etc.,} & \\
\underline{9} & \text{or } .0\dot{\frac{1}{2}} \\
10 & \\
\underline{9} & \\
1 &
\end{array}
$$

Again, let us reduce $\frac{16}{297}$ to a circulate. Annex zeros to the numerator and divide by the denominator: 297 is contained in 160 tenths $\frac{1}{2}$ of a tenth time, with 115 hundredths as a remainder; 297 is contained in 115 hundredths $\frac{1}{3}$ of a hundredth time, with 16 hundredths as a remainder. The remainders

OPERATION.

$$
\begin{array}{r}
297)16.00(.\dot{0}\frac{1}{2}0\dot{\frac{1}{3}} \\
\underline{14\,85} \\
115 \\
\underline{99} \\
16
\end{array}
$$

begin to repeat; hence the quotient figures will repeat and we have $\frac{16}{297}=.\dot{0}\frac{1}{2}0\dot{\frac{1}{3}}$.

Again, reduce $\frac{7}{180}$ to a circulate. We divide the numerator by the denominator: 180 is contained in 70 tenths $\frac{1}{3}$ of a tenth time, with a remainder of 10 tenths or 100 hundredths; 180 is contained in 100 hundredths $\frac{1}{2}$ of a hundredth time, with a remainder of 10 hundredths. Here we observe that the last remainder is the same as the preceding; hence we conclude that $\frac{1}{2}$ will repeat, and we have $\frac{7}{180}=.0\frac{1}{3}0\dot{\frac{1}{2}}$.

OPERATION.

$$180)7.00(.0\tfrac{1}{3}0\dot{\tfrac{1}{2}}$$
$$\underline{60}$$
$$100$$
$$\underline{90}$$
$$10$$

Meaning.—Now what does $.0\dot{\frac{1}{2}}$ signify? First, it is said since $\frac{1}{2}$ of a tenth equals 5 hundredths, that 5 is the repetend, and that the repeating term first appears in the order of hundredths; or, in other words, that $.0\dot{\frac{1}{2}}$ is the same as the mixed circulate $.0\dot{5}$. This view is founded on a wrong conception of the relation of a fraction to the term on the left of it, as may be seen from what has already been explained, and also shows a wrong conception of a repetend. A repetend is simply the term or terms which repeat, and does not concern itself with their place or order value. Thus the expression $.\dot{3}$ does not mean that 3 *tenths* repeats, but simply that the term 3 repeats; hence $.0\dot{\frac{1}{2}}$ does not mean that $\frac{1}{2}$ of a tenth, or 5 hundredths repeats, but simply the expression $0\frac{1}{2}$. It is true that $.0\dot{\frac{1}{2}}$ *equals* $.0\dot{5}$; but it does not mean precisely the same thing, neither is anything gained in general by reducing it to that form. This will appear in some of the subsequent expressions.

Second, it has been said that the expression $.0\dot{\frac{1}{2}}$ represents an absurdity. This cannot be, unless the fraction occurring in a decimal place, with no repeating part, is also absurd. It will be admitted that $.2\frac{1}{2}$ is a legitimate decimal expression; and, if this is so, then it is conceivable that such an expression may be repeated. This conception certainly involves nothing absurd; and hence the expression of this idea, or $.2\frac{1}{2}2\frac{1}{2}2\frac{1}{2}$ etc., seems to be entirely legitimate. This gives rise to the expression $.2\dot{\frac{1}{2}}$; and if the form $.2\dot{\frac{1}{2}}$ is legitimate, so also is $.0\dot{\frac{1}{2}}$. The expression $.0\frac{1}{2}0\frac{1}{2}0\frac{1}{2}$ etc., is a legitimate conception; and hence the abbreviated expression of it, $.0\dot{\frac{1}{2}}$, cannot represent an absurdity.

But, admitting the correctness of the conception, does *one point* express the idea that the entire expression $.0\dot{\frac{1}{2}}$ repeats? It is clear that it will not do to use two dots, since that would indicate that there were two orders of the decimal scale occupied, which cannot be unless such expressions as $2\frac{1}{2}$, $4\frac{1}{3}$, are regarded as occupying tens and units place, which no one would claim. It may be asked, however, whether the point should be placed over the 0 or the $\frac{1}{2}$; thus, $.\dot{0}\frac{1}{2}$ or $.0\dot{\frac{1}{2}}$, and this it is difficult to decide. Both characters occupy tenths place, so that it would seem to indicate the same thing when placed over either character. There is one consideration which inclines one to place the point over the fraction. If we place it over the 0, it might be understood that the 0 repeats and not the $\frac{1}{2}$; but if placed over the $\frac{1}{2}$, it must indicate the repetition of both, since the $\frac{1}{2}$ cannot be repeated without using the 0 in order to locate it. To prevent ambiguity, it would probably be better to place it over both, or rather partly between them; thus, $.\dot{0}\frac{1}{2}$. In the case of expressions occupying more than one place, the meaning will be apparent if the point is placed over the beginning and end of the expression; thus, $.\dot{0}\frac{1}{2}0\dot{\frac{1}{3}}$.

Value.—These forms can be readily reduced to common fractions, and their value thus expressed. A few examples will illustrate the method. Thus, reduce $.0\dot{\frac{1}{2}}$ to a common fraction. Let F represent the fraction equal to $.0\dot{\frac{1}{2}}$. Multiplying the equation by 10, we have $10F=\frac{1}{2}$ of a unit and the circulate $.0\dot{\frac{1}{2}}$. Subtracting the first equation from the second, we have $9F=\frac{1}{2}$, whence $F=\frac{\frac{1}{2}}{9}$, or $\frac{1}{18}$.

OPERATION.

$$\begin{aligned} \text{Let } F &= .0\tfrac{1}{2}0\tfrac{1}{2} \text{ etc.} \\ 10F &= \tfrac{1}{2}.0\tfrac{1}{2} \text{ etc.} \\ \hline 9F &= \tfrac{1}{2} \\ F &= \frac{\frac{1}{2}}{9} = \tfrac{1}{18}. \end{aligned}$$

Again, reduce $.\dot{0}\frac{1}{2}\dot{0}\frac{1}{3}$ to a common fraction. Let F represent the fraction equal to $.\dot{0}\frac{1}{2}\dot{0}\frac{1}{3}$. Multiplying by 100, we have 100F equal to $\frac{1}{2}$ of a ten, $\frac{1}{3}$ of a unit, and the circulate $.\dot{0}\frac{1}{2}\dot{0}\frac{1}{3}$; subtracting the first equation from the second, we have 99F equal to $\frac{1}{2}$ of a ten and $\frac{1}{3}$ of a unit, whence F equals $\frac{1}{2}$ of a ten and $\frac{1}{3}$ of a unit divided by 99; multiplying both numerator and denominator by 6, we have $\frac{32}{594}$, since 6 times $\frac{1}{3}$ of a unit equals 2 units, and 6 times $\frac{1}{2}$ of a ten equals 3 tens.

OPERATION.

$$\text{F}=.0\tfrac{1}{2}0\tfrac{1}{3}0\tfrac{1}{2}0\tfrac{1}{3}\text{ etc.}$$
$$100\text{F}=0\tfrac{1}{2}0\tfrac{1}{3}.0\tfrac{1}{2}0\tfrac{1}{3}\text{ etc.}$$
$$99\text{F}=0\tfrac{1}{2}0\tfrac{1}{3}$$
$$\text{F}=\frac{0\frac{1}{2}0\frac{1}{3}}{99}\times\frac{6}{6}=\frac{32}{594}$$
$$=\frac{16}{297}.$$

Again, reduce $.0\frac{1}{2}\dot{0}\frac{1}{3}\dot{0}\frac{1}{5}$ to a common fraction. Let $\text{F}=.0\frac{1}{2}\dot{0}\frac{1}{3}\dot{0}\frac{1}{5}$. Multiplying by 10, we have 10F equal to $\frac{1}{2}$ of a unit, and $.\dot{0}\frac{1}{3}\dot{0}\frac{1}{5}$; multiplying again by 100, we have 1000F equal to $\frac{1}{2}$ of a hundred, $\frac{1}{3}$ of a ten, $\frac{1}{5}$ of a unit, and the circulate $.\dot{0}\frac{1}{3}\dot{0}\frac{1}{5}$; subtracting 10F from 1000F, we have $\frac{1}{2}$ of a hundred, $\frac{1}{3}$ of a ten, $\frac{1}{5}$ of a unit, minus $\frac{1}{2}$ of a unit, or reducing the $\frac{1}{3}$ of a ten to units, and subtracting the $\frac{1}{2}$ of a unit, we have 990F equal to $\frac{1}{2}$ of a hundred and $3\frac{1}{30}$ units; whence $\text{F}=\frac{0\frac{1}{2}03\frac{1}{30}}{990}$; multiplying both terms of the fraction by 30, we have $\frac{1591}{29700}$, since 30 times $3\frac{1}{30}$ equals 91, and 30 times $\frac{1}{2}$ of a hundred equals 15 hundred.

OPERATION.

$$\text{F}=.0\tfrac{1}{2}\dot{0}\tfrac{1}{3}\dot{0}\tfrac{1}{5}$$
$$10\text{F}=0\tfrac{1}{2}.\dot{0}\tfrac{1}{3}\dot{0}\tfrac{1}{5}$$
$$1000\text{F}=0\tfrac{1}{2}0\tfrac{1}{3}0\tfrac{1}{5}.\dot{0}\tfrac{1}{3}\dot{0}\tfrac{1}{5}$$
$$990\text{F}=0\tfrac{1}{2}0\tfrac{1}{3}0\tfrac{1}{5}-0\tfrac{1}{2}=0\tfrac{1}{2}03\tfrac{1}{30}$$
$$\text{F}=\frac{0\frac{1}{2}03\frac{1}{30}}{990}\times\frac{30}{30}=\frac{1591}{29700}$$

These forms are more curious than practical, and were presented at first merely as puzzles. The discussion here given grew out of the fact that their correctness has been questioned.

Part V.

DENOMINATE NUMBERS.

I. Nature of Denominate Numbers.

II. Measures of Extension.

III. Measures of Weight.

IV. Measures of Value.

V. Measures of Time.

VI. The Metric System.

CHAPTER I.

NATURE OF DENOMINATE NUMBERS.

THE numerical idea is practically the product of two factors —Mind and Matter. We begin by numbering objects, or with *concrete numbers;* we then withdraw the numerical idea from these objects, and obtain pure or *abstract numbers.* The child's first numbers are invariably concrete; from these it passes to number in the abstract. Subsequently, in the effort to apply the numerical idea to quantities not existing in individual forms, there arises a third class of numbers called *Denominate Numbers.* The nature of these numbers, which seems not to have been very clearly apprehended by arithmeticians, will be discussed somewhat in detail.

Nature, regarded as *how many* and *how much,* gives rise to two kinds of quantity—quantity of *magnitude* and quantity of *multitude,* called also *discrete* and *continuous* quantity. These two classes of quantity are entirely distinct, as may be seen by the manner in which they are primarily estimated. Discrete quantity is immediately estimated as *how many;* continuous quantity is primarily estimated as *how much.* Thus, we say *how many* apples, *how many* trees, *how many* birds, etc., not *how much* apples, *how much* trees, *how much* birds, etc. On the other hand, we say *how much* money, *how much* land, *how much* time, *how much* do you weigh, etc.; not *how many* money, *how many* land, etc. When, however, we fix upon some unit of measure, these latter quantities can also be expressed numerically; thus, we may say *how many* dollars, *how many* acres, *how many* pounds do you weigh, etc. In this manner, quantity of magnitude, the *how much*, becomes definitely apprehended as the *how many*. We fix upon a unit of measure, and estimate continuous quantity by comparing it with this unit as a standard. This comparison leads to a numerical apprehension of the quantity considered; we speak of it as so many of the unit of measure. Quantity of magnitude is thus estimated as quantity of multitude; the *how much* is reduced to and regarded as the *how many*. In this manner arises a distinct class of numbers called Denominate Numbers.

A Denominate Number is thus seen to be *a numerical expression of quantity of magnitude*. Quantity of multitude exists primarily as units; in quantity of magnitude we fix upon some particular portion of the quantity as a unit of measure, and estimate the quantity by the number of times it contains this unit. From this conception of the origin and nature of a denominate number, we are prepared to give a scientific definition of it.

Definition.—A Denominate Number is a numerical expression of quantity of magnitude; or, A Denominate Number is a numerical expression of continuous quantity; or, A Denominate Number is a number of units of quantity of magnitude; or, A Denominate Number is a number in which the unit is a measure of a quantity of magnitude; or, A Denominate Number is a number in which the unit is a measure. In using the last definition, we would of course attach a special meaning to the term *measure*. It may also be remarked that in each definition the expression *continuous quantity* may be substituted for *quantity of magnitude*.

The quantities thus considered are *Time*, *Weight*, *Value*, *Length*, etc., which are called quantities of magnitude. The term magnitude, meaning literally size, extent, was primarily applied to quantities occupying space; that is, to something possessing length, breadth, and thickness. In this primary sense it does not include weight, value, and time, since these have no size or extent—no length, breadth, or thickness. The signification of the term, however, has become gradually enlarged, until it now includes every kind of continuous quantity. It was used in this sense, I think, as early as the time of Euclid.

Unit of Measure.—Quantity of magnitude is numerically expressed, as already stated, by comparing it with some fixed quantity of the same species used as a standard of comparison. This standard of comparison is called the *Unit of Measure.* The Unit of Measure is a definite portion of the particular kind of quantity considered. Thus, in weight we take some definite weight as the unit, and estimate the entire weight in numbers by its relation to the unit. The same is done in time, value, length, etc. The unit of measure thus becomes the basis of all these quantities, the quantities themselves being definitely conceived only as we have a definite idea of the unit.

The units of measure by which these quantities are estimated do not exist in nature; they are agreed upon by mankind, and are therefore artificial. They constitute a distinct class of concrete units, and give rise to a distinct class of Concrete Numbers. It is thus seen that there are two classes of concrete numbers,—one in which the unit is *natural*, and another in which it is *artificial.* Thus in 4 *trees*, the unit *tree* is found in nature, and is therefore a natural unit; in 4 *pounds*, the unit *pound* is not found in nature, but is fixed by man, and is therefore artificial. This consideration leads us to another definition of denominate numbers; thus,—A Denominate Number is a concrete number in which the unit is artificial.

The terms *natural* and *artificial,* as here used, refer to the quantities regarded as units and not merely as objects. Thus, in the number 4 *knives*, the knife, as an object, is not found in nature; it is a work of art and therefore artificial. But though artificial as an object, as a unit of measure it is natural. It is, therefore, entirely correct to speak of artificial and natural units, and consequently to define a denominate number as a number of artificial units. The definition previously given, however, is regarded as better.

Quantities.—The quantities of magnitude, or continuous quantities, which give rise to denominate numbers, are of several different kinds. A scientific classification of these quantities is as follows; 1. Value; 2. Weight; 3. Space; 4. Time. The term *extension* is more frequently used than space; we generally speak of *measures of extension* rather than *measures of space.* Space includes several distinct forms of extension; length, surface, volume, etc.; and hence a popular classification, and one a little more convenient in practice, is the following: 1. Value; 2. Weight; 3. Length; 4. Surface; 5. Volume; 6. Capacity; 7. Angles; 8. Time.

The unscientific manner in which the subject of denominate numbers has been presented, has led to many incorrect ideas concerning them. Some writers have considered them under the head of "Weights and Measures;" seeming not to know that the measure of the force of gravity is just as much a measure as the measure of length. One writer says they may be divided into three general classes—Currency, Measure and Weight—seeming not to understand that currency is a measure of value, and weight a measure of gravity. It is hoped that the views here presented will lead arithmeticians and teachers to a more correct and philosophical view of this subject.

It is a peculiarity of these numbers that they have both an abstract and a concrete signification. The denominate numbers refer to time, weight, value, etc: these things are not tangible, material things. Time is nothing that you can touch or see, and the same is true of value and weight. Length, surface, and volume are the abstract quantities of geometry. Concrete things possess value and weight; but the value and weight are no more concrete than are the length, surface, or volume. Denominate numbers are therefore not concrete, in the sense that the units are material things. They are concrete, however, in the sense that the number is associated with some particular unit. The word concrete, from *con*, with, and *cresco*, I grow, means literally growing or united together. A number is properly concrete when it is associated with something numbered, even though the thing itself may be abstract in its nature. It is not the character of that which is numbered, but the fact of the association of a number with it, that makes the number concrete. Thus, the number *four* used independently of any object is abstract, but if associated with some object, as *boy*, *yard*, *pound*, etc., we have the concrete numbers 4 *boys*, 4 *yards*, etc.

The Standard Units.—The Units of Measure are of as many distinct classes as there are distinct kinds of continuous quantity to be measured. There are logically four different kinds of such quantity, and consequently there are four distinct classes of units of measure. These units, originating by chance, were indefinite and variable, and, in time, were found unsuited to the purposes of a civilized people. Science then took the matter in hand and began to establish standard units, having definite values derived from some invariable element of nature by which we would be able to reconstruct the measures at any time if destroyed. These standard units were so related to each other that, having fixed one of them, all the others could be derived from it. The fundamental unit agreed upon was that of *length*. To obtain a standard unit of length was now the question. The French endeavored to fix it by ascertaining the distance from the equator to the poles, and taking a definite part of this distance.

This was done by Delambre and Méchain, who measured an arc of the meridian between Dunkirk and Barcelona, and gave the standard unit of length, called the *Meter*. The English fixed their standard unit of length by finding the length of a pendulum which vibrates seconds. The latter method is regarded as the most convenient in practice, though it is criticised by the French as being dependent on two elements foreign to length,—that is, gravity and time. From the unit of length, however obtained, all the other units, except that of time, may be derived. It will be noticed that, in the English method, time is made the basis of the system.

The *Standard Unit of Time* is the *Day*. This is determined by the revolution of the earth upon its axis.

The *Standard Unit of Value*, in the United States, is the *Dollar*. It is determined by the weight of the metal used for money. In English Money the standard is the *Pound*, determined in a similar manner.

The *Standard Unit of Weight* is the *Troy Pound*. It is determined by taking a certain number of cubic inches of distilled water at a given temperature, the barometer being at a certain altitude. The Avoirdupois pound is derived from the Troy pound, by taking 7,000 Troy grains. The Apothecaries' pound is the same as the Troy pound.

The *Standard Unit of Length* is the *Yard*. It is determined by the length of a pendulum which vibrates seconds in a vacuum at the level of the sea, in the latitude of London. Such a pendulum is divided into 391,393 equal parts and 360,000 of these parts taken for the yard.

The *Standard Unit of Surface* is the *Square Yard* for ordinary measurement, and the *Acre* for land. The standard unit of *Volume* is the *Cubic Yard* for ordinary measurement, and the *Cord* for wood. These are derived from the unit of length.

The *Standard Unit of Capacity* is the *Gallon* for fluids, and the *Bushel* for dry substances. These are also determined from the unit of length, each measure consisting of a certain number of cubic inches.

The *Standard Unit of Angular Measure* is the *Right Angle*, or, in practice, one degree of a circle.

These standard units, as stated above, are so related to each other that, having determined one, all the others may be derived from it. Time is the basis of the English system. We first find the Unit of Time from the revolution of the heavenly bodies, and dividing it sufficiently far we obtain the *second.* Having the second, we can obtain the unit of length by ascertaining the length of a seconds pendulum and taking a definite part of it. Having the unit of length, we readily obtain the units of area, volume, and capacity. The standard unit of weight is obtained by taking a number of cubic inches of water. The unit of value is a weight of gold or silver, and can thus be traced back to its origin in Time.

Scales.—These standard units are divided into smaller units, each receiving a name and being used as a unit of measure, and these are again subdivided in a similar manner. Multiples of the standard units are also used as new measures, multiples of these in the same way, the series being continued as far as convenient. This gives a series of measures for the estimation of the same kind of quantity, forming a scale of numbers. Any number expressed in two or more terms of such a scale, constitutes what is called a *Compound Number.* A Compound Number, therefore, consists of several denominate numbers of the same kind of quantity.

Since the different standards of comparison, their multiplication and division, originated at different times and under different circumstances, it is natural that these scales should be irregular and without system. Sometimes the scale of increase is by twos, fours, etc.; sometimes by fours, twelves, etc.; and again by 12, 3, $5\frac{1}{2}$, etc. This irregularity will be most clearly seen by comparing any scale of compound numbers with the decimal scale. Take, for instance, the scale of English Money and write it beside the decimal scale; thus,

T.	h.	t.	u.		£	s.	d.	qr.
1	1	1	1		1	1	1	1.

In the decimal scale the second unit is *ten* times the first, the third is *ten* times the second, etc. In the scale of English Money, the second unit is *four* times the first, the third is *twelve* times the second, the fourth is *twenty* times the third. The same irregularity obtains among all the other scales of denominate numbers.

This irregularity of scale is a serious defect of our measures of value, weight, etc., viewed from either a scientific or a practical standpoint. Science dictates that the multiples and divisions of the standard units should be uniform and correspond to the scale of notation, which with us would make the scales decimal. This would enable us to add, subtract, multiply, and divide compound numbers just as we do abstract numbers. France has adopted this method in all its measures, and the United States in its currency. This method has many very important advantages; the only objection to it arises from the *decimal* base of our system of notation, since the simple fractional parts, *thirds*, *fourths*, and *sixths*, are not aliquot parts of ten. With a system of notation whose base was twelve, the method would be much more convenient in practice.

CHAPTER II.

MEASURES OF EXTENSION.

THE Measures of Extension are of three kinds, measures of Length, of Surface, and of Volume or Capacity. This division arises from the fact that extension possesses but three elements; length, breadth, and thickness. To these, however, must be added Angular or Circular Measure, which is the degree of divergence of two lines, or the length of an arc of a circumference used to indicate this divergence.

The standard units of this measure were originally derived from the natural objects of the material world. In their origin they followed the general law of mental and scientific development—from the concrete to the abstract. Thus originated the *foot*, the *cubit*, the *span*, the *fathom*, the *barleycorn*, the *hair-breadth*, and other measures taken from parts of the human body, or from natural objects which possess a certain mean value sufficiently definite to answer the purposes of a rude society. The same is true for all the other measures in denominate numbers. Among the measures of weight we have the *grain*, originally a grain of wheat; the *pennyweight*, which was the weight of the English penny, etc. For measures of value some nations use *cattle;* others, *pigs;* the Icelanders, dried fish; the American Indians, the skins of animals. The first measures of time were derived from the revolutions of the heavenly bodies: thus *month*, derived from *moon-eth*, was originally the time measured by the revolution of the moon. The subject being interesting and instructive, will be treated somewhat in detail. The principal authorities followed are Spencer and the popular encyclopedias.

Nearly all of the original measures of length were taken from the parts of the human body. The Hebrew *cubit* was the length of the forearm from the elbow to the end of the middle finger; the smaller Scriptural dimensions are expressed in *hand-breadths* and *spans*. The Egyptian *cubit*, which was similarly derived, was divided into *digits* which were *finger-breadths;* and each finger-breadth was regarded as equal to four *grains of barley* placed breadth-wise. Other ancient measures were the *orgyia*, or *stretch of the arms*, the *pace*, the *palm*, etc. So general and persistent was the use of these natural units in the East, that even now some of the Arab tribes mete out cloth by the *forearm*, with the addition of the breadth of the other hand, which marks the end of the measure. The width of the thumb was in like manner added at the end of the yard by English clothiers; and it is not unusual to see women, even in this country, test the number of yards in a purchase by measuring it from the chin to the end of the fingers. The *foot* was used by the Romans, and is still a measure in Europe and America, its length in different places varying not much more than men's feet vary. The height of horses is expressed in *hands*, and the depth of water in *fathoms*, the length of the two arms extended. The *inch* is supposed to be the length of the terminal joint of the *thumb*, as appears in the French language, where *pouce* means both thumb and inch. We have also the inch divided into *barleycorns*, the inch being regarded as equal to three of these.

So completely indeed have these natural dimensions served as the basis of all mensuration, that it is only by means of them that we can form any estimate of some of the ancient distances. For example, the length of a degree on the earth's surface, as determined by the Arabian astronomers, shortly after the death of Haroun-al-Raschid, was fifty-six of their miles. We know nothing of their mile, further than that it was 4,000 cubits; and whether these were sacred cubits or common cubits, would remain doubtful, but that the length of the cubit is given as twenty-seven inches, and each inch defined as the thickness of six barley-grains. Thus one of the earliest measurements of a degree comes down to us in barley-grains. Not only did organic lengths furnish those approximate measures which satisfied men's needs in ruder ages, but they furnished also the standard measures required in later times. One instance occurs in the reign of Henry I., who, to remedy the irregularities then prevailing, commanded that the *ulna*, or ancient ell, which answers to the modern yard, should be made the exact length of *his own arm*.

As civilization advanced, the inaccuracy of such variable measures as the foot, cubit, etc., were felt, and the necessity of adopting more precise standards became apparent. To do this, it was necessary to find, among the objects of nature, a standard perfectly definite, and at the same time invariable and accessible to all mankind. To obtain such an object was a matter of no inconsiderable difficulty. In fact, nature presents only two or three elements which, with the aid of a profound science and a refined knowledge of the arts, can be made subservient to the purpose, and none at all without such aid. The earth is nearly a solid of revolution, and its form and absolute magnitude are presumed to remain the same in all ages; hence the distance from the equator to the poles is an invariable quantity, and some definite part of this distance, if exactly ascertained, might be taken as a standard unit of length. The force of gravity at the earth's surface is constant at any given place, and is nearly the same at all places under the same parallel of latitude, and at the same height above the level of the sea; hence the length of a pendulum which makes a given number of vibrations in a given time is constant, and might be used to determine a standard unit of length.

These two elements, the length of a degree of the meridian, and the length of a seconds pendulum, are the only ones furnished by nature which have yet been used as a basis of a system of measures. One or two others have been suggested, as the height through which a body falls in a second of time, and the perpendicular height through which a barometer must be carried till the mercurial column sinks a determinate part, for example, one-thirtieth of its own length; but these distances are not so susceptible of being accurately determined as the terrestrial degree, or the length of the seconds pendulum. Mouton, an astronomer of Lyons, about 1670, proposed as a standard, a *geometrical foot*, of which a degree of the earth's circumference should contain 600,000; and remarked that a pendulum of this length would make $3959\frac{1}{5}$ vibrations in a half hour. In 1671, Picard proposed a similar method; and Huygens first suggested the pendulum as the unit or standard of measures. No attempt was made to establish a regular system of measures until the time of the French Revolution, when a system of weights and measures, referred to the terrestrial degree, and accommodated to our arithmetical scale, was adopted in that country.

English Standard Measures.—The English standard unit of length is the yard, which, in the reign of Henry I., was determined by the length of the king's arm. An act of Edward II., 1324, provides that the length of 3 barleycorns, round and dry, shall make an inch, 12 inches a foot, etc. The difficulty of determining how much of the end of the grain should be removed to render it "round" makes this standard rather indefinite. No record exists, however, of the actual construction of standard units based upon the use of barleycorns. In time, for the purpose of securing some uniformity among the ordinary measures, certain standards were placed in the Exchequer, with which all rods were required to be compared before they were stamped as legal measures. The oldest of the standards in existence dates from the reign of Henry VII., but it has long been disused. There was another similar rod of the same date, called an *ell*, though it was not established as a legal measure, but was conventionally regarded as equal to a yard and a quarter. That which, till the year 1824, was considered as the legal standard, was a brass rod of the breadth and thickness of about half an inch, placed there in the time of Elizabeth. This standard was illy fitted for the purpose designed. "A common kitchen poker, filed at the end in the rudest manner, by the most bungling workman, would make as good a standard. It has been broken asunder and the two pieces dovetailed, but so badly that the joint is nearly as loose as that of a pair of tongs."

In the year 1742, some Fellows of the Royal Society, and members of the Academy of Sciences at Paris, proposed to have accurate standards of the "measures and weights" of both nations made and carefully examined, in order that the results of the scientific experiments in England and France might be correctly compared. The committee who undertook the matter, besides the standard in the Exchequer, found some others which were considered of good, if not of equal authority. At Guildhall they found two standards of length, which were only two beds or matrices, one of a yard, the other of an ell, cut out of the edges of a brass bar, like that of the Exchequer. Another, kept in the Tower of London, was a solid brass rod, about seven-tenths of an inch square, and 41 inches long, on one side of which was a yard divided into inches. Another, belonging to a clockmakers' company, derived from the Exchequer in 1671, was a brass rod of eight sides, on which the length of the yard was expressed by the distance between two pins or small checks. The committee selected the standard in the Tower, and Mr. George Graham, a celebrated clockmaker, in 1742, laid off from it, with great care, the length of the yard on two brass rods which were then sent to the Academy of Sciences at Paris; on these, in like manner, was set off the measure of the Paris half toise. One of these was kept at Paris, the other was returned to the Royal Society, where it still remains; but unfortunately it was not stated at what temperature the toise was set off, so that the comparison is now of little value.

In 1758, a committee of the House of Commons recommended that a rod which had been made at their order, by Mr. Bird, from that of the Royal Society, and marked "Standard Yard of 1758," should be declared the legal standard of all measures of length. In the following year another committee was formed on the subject, which concurred in the Bird standard; and authorized Mr. Bird to make a copy of the former rod, which he completed in 1760. No further action was taken by the government until 1824, when a very thorough revision was attempted.

Stimulated by the scientific efforts of the French philosophers, the English turned their attention to the establishment of an invariable standard unit of length, and selected as a basis the seconds pendulum at London. The length of such a pendulum had been determined as early as 1742, by George Graham, to be 39.13 inches, and used for the construction of a standard yard. Reports were made in 1816, 1818, and 1820, to the House of Commons, based on experiments and comparisons, in which Wollaston, Dr. Young, Capt. Kater, and Prof. Playfair, took a prominent part. This led to the adoption of the imperial measures and standards under George IV., which took effect January 1, 1826: these standards were retained by the enactments under William IV., which took effect January 1, 1836. In the imperial measures, the yard copied from the standard of 1760 was to be of brass, and measured at the temperature of 62° F., while its length was further defined by declaring that the pendulum beating seconds of mean time in the latitude of London at the temperature named, in a vacuum at the level of the sea, should be 39.1393 inches of the above standard. From this standard measure of length, all the other measures, of weight, capacity, etc., were also established.

Soon after the standards were prepared, they were destroyed by the burning of the Houses of Parliament, 1834; but fortunately the Astronomical Society had procured a most carefully prepared copy of the imperial standard yard, and the mint was in possession of an exact copy of the pound, so that it was possible to reproduce the lost standards with great precision.

In 1838 a commission was appointed, of which Airy, Baily, Herschel, Lubbock, and Shepherd were members, which, after a very thorough investigation of the matter, reported in 1841, that, since the passage of the act of George IV., several elements of reduction of the pendulum experiments, on which some of its provisions were based, had been found to be doubtful or erroneous, there having been defects in the agate planes of the pendulum used by Capt. Kater, and errors in finding its specific gravity, and in reductions for buoyancy of the air and for elevation above the level of the sea. They concluded that the course prescribed in the act would not produce the original yard; that the condition that the yard was to be a certain brass rod was the best that could be adopted; and that, by the aid of the Astronomical Society's scale, and a few other highly accurate copies known, the standard could be restored without sensible error. Mr. Baily was selected to prepare the new standard, with five copies of the preceding on which to base his comparison; and on his death in 1844, Mr. Sheepshanks continued the necessary observations, executing himself, in the course of this labor, about 200,000 micrometric measurements. He prepared several standard copies, each being a square inch bar, of a bronze consisting of copper with a small percentage of tin and zinc, 38 inches in length, with half-inch wélls sunk to the middle of the bar, one inch from each end, in which the lines defining the yard are drawn on gold plugs. Six of these were finally selected and reported by the commissioners in March, 1854; and one of these, marked "Bronze 19," was selected as the parliamentary standard yard, the remaining five being deposited, along with copies of the standard of weight, with as many public institutions and scientific bodies. These standards were legalized in July, 1855; and provisions made that in case of loss of the parliamentary copy, the standard should be restored by comparison of the other selected copies, or such as might be available. From this statement it is seen that the latest verdict of science is adverse to the practicability of basing a system of measures on any invariable natural unit of length.

The measures of the American colonies were the same as those of the mother country at the corresponding period. Variations naturally grew up in the different colonies, and the several measures in use being adopted with little or no change when they became states, the discrepancies continued to exist. On the 3d of March, 1817, John Quincy Adams was commissioned by a resolution of the Senate, to examine the subject of the weights and measures of the United States, and also to consider the desirableness of adopting the French system, or one similar to it. The standards employed in the various custom-houses were examined and carefully measured during 1819 and 1820, under his direction, and in a report published in 1821 he showed that very considerable discrepancies existed between the measures of the several states, and often within the same state. After a careful review of the French system, he reported unfavorably to its adoption, on account of the difficulty of the change and the essential inconvenience of a decimal system.

On the 29th of May, 1830, the Senate directed a new comparison of the weights and measures in use at the different custom-houses. This examination was made by Prof. Hassler, who found that, though considerable discrepancy existed, the mean value corresponded closely with the English standards verified in 1776. Under his supervision accurate copies of the received standards of weights and measures were supplied to all the custom-houses; and by act of Congress, June 14, 1836, the Secretary of State was directed to have sent to the governors of all the states a complete set of all standards, for the use of the several states. These, as well as accurate balances for adjusting the weights, were supplied; and the statutory standards of every state have been made to conform to the standards so furnished. No enactments were made concerning the old English standards of length, which have come down to us, as they were necessarily in force unless changed by legislative enactment.

It is to be noticed that the American yard was taken from a scale made for the United States by Troughton, which was supposed to be identical with the old standard, and with the Astronomical Society's scale, but which had never been directly compared with either. When this comparison was made with the bronze bar No. 11, which had been presented to the United States by the British Government, the yard on the Troughton scale was found to be nearly $\frac{1}{1000}$ of an inch too long; and hence all the copies furnished to the states are subject to that minute correction, since the British yard is unquestionably the only authentic representative of the old standard from which our measures are derived. In 1866 Congress authorized the use of the metric system, but the present indications are that it will be very tardily introduced.

Measures of extension, as already stated, are of three distinct classes: measures of Length, measures of Surface, and measures of Volume or Capacity. Measures of Length are of several different kinds; the common Long Measure, Surveyors' Measure, etc. Measures of Surface include the ordinary Surface Measure, and Surveyors' Square Measure. Measures of Volume include the ordinary Cubic Measure, and Measures of Capacity. Measures of Capacity embrace those of Liquids and Dry Substances. A few facts in addition to those already given will be stated.

Long Measure.—Long Measure is the measure of length, applied to the measurement of the length, breadth, or thickness of objects; also to heights and distances. The unit, as already explained, is the *yard*, which is identical with the Imperial yard of Great Britain, divisions and multiples of which give an irregular scale of derived units. These units were nearly all originally derived from parts of the human body. Thus, *foot* is from the human foot; *yard* was a rod or shoot; *rod* is from a measuring stick or rod; *furlong* is from *fur*, furrow, and *long*, meaning long, or the length of a furrow; *mile* is from *mille passuum*, a thousand paces; *span* is the space measured by the thumb and little finger extended; *cubit*, the forearm; *fathom*, the length of the two arms extended. The *inch* is supposed by some to be derived from the terminal joint of the thumb, though late etymologists get it from *uncia*, the twelfth part. The inch was formerly divided into three equal parts, called *barleycorns*, the length of a grain or kernel of barley.

The *geographic mile* is equal to 1 minute of one of the great circles of the earth, hence it equals $\frac{1}{60}$ of $\frac{1}{360}$ of the circumference of the earth, which equals about 1.15 statute miles. The *knot*, used in measuring distances at sea, is equivalent to a geographic mile. The English mile is the same as that of the United States. The German short mile equals 6857 yards, or about $3\frac{9}{10}$ statute miles; the German long mile equals 10125 yards, or about $5\frac{3}{4}$ statute miles; the Prussian mile equals 8237 yards, or about $4\frac{7}{10}$ statute miles.

A *degree* of longitude at any point is $\frac{1}{360}$ of the circle passing through the latitude of that point; and as these circles diminish as we pass from the equator, the degrees of longitude will diminish. Thus, at the equator, the length of a degree of longitude is about $69\frac{1}{6}$ statute miles; at 25° of latitude, $62\frac{7}{10}$ miles; at 40° of latitude, 53 miles; at 42°, $51\frac{1}{2}$ miles; at 49°, $45\frac{1}{2}$ miles; at 60°, $34\frac{7}{12}$ miles, etc. A degree of latitude also varies, being 68.72 miles at the equator; from 68.9 to 69.05 miles in middle latitude; and from 69.30 to 69.34 miles in the polar regions.

Surveyors' Linear Measure is used by surveyors and engineers in measuring the dimensions of land, distances, etc. The unit is the *chain*, called Gunter's chain, from Edmund Gunter, the reputed inventor, an English mathematician, born 1581, died 1626. It is 4 rods, or 66 feet in length, and is divided according to the decimal system into 100 equal parts called *links*. This division reduces all the calculations to the decimal system, and thus greatly simplifies the operations.

The denomination *rods* is seldom used by surveyors, distances being represented in chains and links. Since each link is $\frac{1}{100}$ of a chain, the number of links is generally expressed as a decimal; thus 5 chains and 47 links are written 5.47 chains. Engineers generally use a chain 100 feet long, containing 120 links, each 10 inches in length. Mariners' Measure is used by seamen in measuring distances, the depth of the sea, etc. The old method of Cloth Measure is practically obsolete, the division of the yard being into *halves*, *quarters*, etc., and cloth being thus sold instead of by the *nail* or inch. At the custom-houses, the yard is divided into tenths, hundredths, etc.

Surface Measure.—Surface or Square Measure is used in measuring surfaces, as land, boards, amount of painting, papering, plastering, paving, etc. The term *Perch* is from the French *perche*, a pole; *rood* is supposed to be a corruption of *rod; acre* was, primarily, an open plowed or sowed field. The unit for land is the acre; for other surfaces it is usually the square yard. The *Perch* is a surface equivalent to a square rod. The *Rood* is less used than formerly. A square piece of land, measuring 209 feet, or about 70 paces, on each side, equals very nearly 1 acre.

Surveyors' Square Measure is used by surveyors in computing the area or contents of land. The *perch* and *rood* are not so much used as formerly, the contents of land being commonly estimated in square miles, acres, and hundredths. Government lands are divided by parallels and meridians into townships, which contain 36 square miles or sections, and each section is subdivided into quarter-sections. Hence 640 acres make a *section*, and 160 acres a *quarter-section*. A *hide* of land, which is spoken of by ancient writers, is 100 acres.

Cubic Measure.—Cubic or Solid Measure is used in measuring things which have length, breadth, and thickness. The fundamental unit seems to be the *cubic foot*. The other units are not in a regular scale, but are parts or multiples of the cubic foot. The old unit for measuring timber was the *ton.* Round timber, when squared for use, is supposed to lose $\frac{1}{5}$; hence a ton of round timber is said to contain such a quantity of timber in its rough or natural state as, when hewn, will make 40 cubic feet, and is supposed to be equal in weight to 50 cubic feet of hewn timber. Timber is now generally sold by board measure, the ton being nearly obsolete. A *cord* of wood is a pile 8 feet long, 4 feet wide, and 4 feet high. A cord foot is a part of this pile 1 foot long; it equals 16 cubic feet.

A *perch* of *stone* or *masonry* is $16\frac{1}{2}$ feet long, $1\frac{1}{2}$ feet wide, and 1 foot high; it contains $24\frac{3}{4}$ cubic feet. A cubic yard of earth is called a *load.* A *square* of earth is a cube measuring 6 feet on a side, and contains 216 cubic feet. In civil engineering, the unit is the cubic yard, to which all estimates for excavations and embankments are reduced. The measurements are taken with a line divided into feet and decimals of a foot. A *Register Ton* is the standard for estimating the capacity or tonnage of vessels, and is 100 cubic feet. A *Shipping Ton*, used in estimating cargoes, in the United States is 40 cubic feet; in England, 42 cubic feet.

Liquid Measures.—Liquid Measures are used for measuring all kinds of liquids. They are of three classes: Wine Measure, Beer Measure, and Apothecaries' Fluid Measure. Wine measure is now used for measuring nearly all kinds of liquids. It was called wine measure because it was used to measure wine, instead of beer, which was measured by another measure.

The standard unit of wine measure is the *gallon.* It is intended to represent the old wine gallon of 231 cubic inches, but is defined as containing 58,372.2 grains of distilled water at its maximum density, weighed in air of the temperature of 62° F., and barometric pressure of 30 inches. This is identical with the old Winchester gallon of England, so called from the standard having been formerly kept at Winchester, England. The Imperial gallon, adopted by Great Britain in 1836, is defined as containing 10 pounds avoirdupois of distilled water.

at a temperature of 62° Fahrenheit, the barometer standing at 30 inches. It contains 277.274 cubic inches. The American gallon is thus 0.83311 of the Imperial gallon, or about 6 of the former equal 5 of the latter.

There were in England, from 1650 to 1688, three different measures of the wine gallon. The one of general usage contained 231 cubic inches to the gallon. Another, the customary standard at Guildhall, supposed to be of the same capacity, was found by measurement to contain only 224 cubic inches. A third, the real and legal standard, preserved at the Treasury, contained 282 cubic inches. The corn gallon differed from any of these, being 268.6 cubic inches. Some suppose the gallons of 231 and 282 cubic inches to have originated under separate enactments, the latter from one of Henry VII., directing that the gallon contain 8 pounds of wheat; but Oughtred holds that the larger or beer gallon was allowed for liquids which yield froth, as beer, etc., and the smaller for such liquids, as wine and oil, which do not froth, and thus their exact volume is immediately indicated.

The term *gill* is from Low Latin *gilla*, a drinking glass; *pint* is from the Anglo-Saxon *pyndan*, to shut in, to pen, or from the Greek *pinto*, to drink; *quart* is from the Latin *quartus*, a fourth. The derivation of *gallon* is not clear; in the French, a *galon* is a grocer's box. Barrels and hogsheads are of variable capacity. The values given in the tables are used in estimating the capacity of wells, cisterns, vats, etc. In Massachusetts the barrel is estimated at 32 gallons. A pint of water weighs nearly one pound, hence the old adage, "A pint's a pound, the world around."

Ale or Beer Measure was formerly used in measuring ale, beer, and milk. It was named Beer Measure from its being so extensively used in measuring beer, in distinction from the measure used for wine and oil, etc. The measure was greater than wine measure, as beer was less costly than wine, or, as some have supposed, on account of the frothing of beer. The unit is the *gallon*, which contains 282 cubic inches, or 10.179933 pounds avoirdupois of distilled water. This measure is going out of use; milk and also beer and ale are now generally measured by Wine Measure.

Apothecaries' Fluid Measure is used for measuring liquids in preparing medical prescriptions. *Minim* is from the Latin *minimus, the least*, the minim being the smallest fluid measure used. Several of the other terms are formed by prefixing *fluid* to the terms of Apothecaries' Weight. *Cong.* is the abbreviation of *congius*, the Latin for gallon. O. is the initial of *octarius*, the Latin for *one-eighth*, the pint being one-eighth of a gallon. In estimating the quantity of fluids, 45 drops equal about a fluid drachm; a common teaspoon holds about 1 fluid drachm; a common tablespoon about ½ a fluid ounce; a wine-glass about 1½ fluid ounces; a common teacup about 4 fluid ounces.

Dry Measure.—Dry Measure is used for measuring dry substances, such as grain, fruit, salt, coal, etc. The unit of dry measure is the bushel, which is divided into pecks, quarts, etc. The term *bushel* is derived from a word meaning *box*. The term *peck* is supposed to be a corruption of *pack*, or to be derived from the French *picotin*, a peck. The Imperial bushel of England, by the act of George IV., was defined to contain 8 gallons. It thus contains 80 pounds of distilled water, or 2218.192 cubic inches. The "heaped bushel" of 2815 cubic inches, declared by the same act as a measure for coals, lime, potatoes, fruit, and fish, was abolished in 1835, by an act of Parliament, during the reign of William IV. The Winchester bushel, in use from the time of Henry VII. to 1826, contained 2150.42 cubic inches. The original standard Winchester bushel, as well as the yard, is still preserved in the museum at Winchester.

The unit of dry measure in the United States is the Winchester bushel, the same as the old English standard. Its form is a cylinder, 18½ inches in diameter, and 8 inches deep. Its volume is 2150.42 cubic inches, and it contains 77.627413 pounds avoirdupois of distilled water, at its maximum density, 39.8° F., barometer 30 inches. The New York bushel is declared to contain 80 pounds of distilled water at its maximum density, and is thus identical with the Imperial bushel of Great Britain. The *chaldron*, consisting in some places of 36 bushels, and in others of 32 bushels, is used in some parts of the United States for measuring coal and coke, but is being discontinued here as it has been in England. One half of a peck, or four quarts, is called a *dry gallon*. The chaldron was divided into *vats*, *sacks*, and *bushels*. The coal bushel held 1 quart more than the Winchester bushel. Twenty-one chaldrons made a *score*.

The *cental* is a standard recently recommended by the Boards of Trade in New York, Cincinnati, Chicago, and other large cities, for estimating grain, seeds, etc. Were this standard generally adopted, the discrepancies of the present system of grain dealing would be avoided. Bushels are changed to centals, by multiplying by the number of pounds in one bushel, and dividing the product by 100. The remainder will be hundredths of a cental.

CHAPTER III.

MEASURES OF WEIGHT.

WEIGHT is the measure of the force of gravity. All bodies are attracted towards the center of the earth in proportion to the quantity of matter contained. This influence being a constant force, it was seen that it might be employed in comparing bodies, and determining their relative quantity of matter. Some standard being fixed upon, the relation of bodies to this standard may be expressed numerically, and thus there will arise a system of measures denominated weights.

Measures of Weight, like those of length, originated with natural objects. Seeds seem to have supplied the original unit. The *carat*, used for weighing in India, is a small *bean*. The basis of the English scale of weights is a *grain of wheat*. Henry III. enacted that an *ounce* should be the weight of 640 dry grains of wheat from the middle of the ear. The *pennyweight* was the weight of an English penny, which varied in size until the time of Queen Elizabeth.

The weight of a body is the measure of the force by which it is drawn towards the center of the earth. The determination of weight consists in the comparison of the object to be weighed with some fixed standard. Such a standard could not be precisely defined by written law or oral explanation, or in any way except by the test of muscular resistance. Having a fixed standard, the weight of bodies would be estimated by comparing them with this standard. The comparison of weight was not quite so readily made as that of length, as a balance was necessary, the construction of which required some degree of mechanical knowledge. The balance, or scales, however, is known to have been in existence, in a rude form, from very early times. The Greeks, as appears from the Parian chronicle, believed weights, measures, and the stamping of gold and silver coins to have been alike the invention of Phidon, ruler of Argos, about the middle of the 8th century B. C.

The standards of weight were even less definite than those of length. This is apparent from the use of such units as *stone*, *load*, *last*, etc. Even the term *pound* (*pondus*) implied only weight indefinitely. The *grain*, taken from the grains or corns of wheat, was, as a standard of small weights, about the only denomination of weight that would universally convey anything like a definite idea. A statute of Henry III., in 1266, enacts "that an English penny, called the sterling, round without clipping, shall weigh 32 grains of wheat, well dried and gathered out of the middle of the ear; and 20 pence (pennyweights) to make an ounce, 12 ounces a pound, 8 pounds a gallon of wine, and 8 gallons of wine a bushel of London, which is the 8th part of a quarter."

In some countries, measures of weight seem to have had their origin in the measures of value. Thus, in Latin, to pay money was to weigh it; and nearly all weights are supposed to have had their origin in the practice of weighing specie. The Sicilian pound, which has been adopted in all countries that accepted the Roman system, and the German mark, both fundamental in all European weights, are said to have been originally quantities of silver. England accepted the pound thus derived, and also attempted to give precision to smaller sums of money by weighing them by grains of corn, as already stated. Even as late as the time of Elizabeth, payments seem to have been invariably made by weight; indeed, the whole metrical system of England was derived from money weights. The payments by tale having superseded those by weight, the real origin of weights is liable to be overlooked.

As science was developed, it began to concern itself in the establishment of a fixed system of weights. The object was to find some invariable standard by which such a system could be derived. As there is a constant ratio between the volumes and weights of the same substances when placed in the same physical circumstances, it was seen that standards of weight may be derived from those of length. For example, a cubic inch of distilled water, at the same temperature and under the same atmospheric pressure, will always have the same weight. Advantage has been taken of this property of bodies to connect measures of weight with those of length, and the weight of a given bulk of water at a fixed temperature is now the standard from which all weights are derived.

The establishment of a system of weights early engaged the attention of the English people. In order to arrive at some definite standard, it was declared, in the Great Charter, that the weights should be the same all over England; but no ordinance, perhaps, was ever so ill observed. The old English pound, which is said to have been the legal standard of weight from the time of William the Conqueror to that of Henry VII., was derived from the weight of grains of wheat, as stated above. Henry VII. altered this weight, and introduced the *Troy* pound instead, which was one-sixteenth part, or three-fourths of an ounce, heavier than the Saxon pound. The Troy pound was divided in the same manner as the Saxon pound, into ounces, pennyweights, and grains; but the pennyweight contained only 24 grains, and consequently a grain Troy became a much heavier weight than the grain of wheat. In fact, the pound Troy contains 5,760 grains, while the Saxon pound, which was divided into 7,680 grains, contained only 5,400 Troy grains. The *avoirdupois* pound was introduced by a statute of Henry VIII. Its first object was to weigh butchers' meat in the market, but it gradually came to be used for all kinds of coarse goods or merchandise. Two legal measures of weight were thus established, and have continued to be used in England ever since, and have been also introduced into this country. The standard of these weights was definitely fixed by Act of Parliament in 1824. The standard brass weight of the pound Troy, made in 1758, and then in custody of the Clerk of the House of Commons, was declared to be the Imperial standard Troy pound, and that 7,000 Troy grains shall constitute an avoirdupois pound.

The retention of two different systems of weights was in compliance with the common usages of the country. Mr. Davies Gilbert stated the reasons for it as follows: "The Troy pound appeared to us to be the ancient weight of this kingdom, having, as we have reason to suppose, existed in the same state in the time of Edward the Confessor; and there are reasons, moreover, to believe that the word *Troy* has no reference to any town in France, but rather to the monkish name given to London of Troy Novant, founded on the legend of Brute. Troy weight, therefore, according to this etymology, in fact, is London weight."

It was also enacted that if the standard Troy pound should be lost or destroyed, it was to be restored by a reference to a cubic inch of distilled water, which has been found and is declared to be 252.458 Troy grains at the temperature of 62° Fahrenheit, the barometer being at 30 inches. The weight of a pennyweight Troy is thus to that of a cubic inch of distilled water in such circumstances, as 24 to 252.458, or of 24,000 to 252,458; so that the weight of the cubic inch of distilled water must be conceived to be divided into 252,458 equal parts, and 24,000 of such parts will be the standard pennyweight, or 240 of such pennyweights will be the standard pound.

A committee appointed in 1843 published a report in 1854, in which they determined to take the avoirdupois pound of 7,000 grains as the standard, and to construct the Troy pound from it. They took the brass Troy pounds in the custody of the Exchequer and certain *platinum* pounds in the possession of the Royal Society and others; but the former being found to have gained in weight by oxidation, only the platinum pounds were used. From them a platinum pound was prepared, weighing in vacuo 6999.99845 grains. The form of the weight is a cylinder, with a groove surrounding it a little above the middle of its height, for the insertion of the ivory fork used in lifting it. The weight is enclosed in a mahogany box, the parts of which, when screwed together, cause the weight to be immovable. This box is enclosed in another box, and with the standard yard in a third box, and finally in a stone case, in the vaulted stone room of the Exchequer.

For philosophical purposes and in delicate weighing, Troy weight only is used, and the weight is usually reckoned in grains By this means fractional numbers are avoided, and no ambiguity can arise, as there are no other grains than Troy grains. Dr. Kelly, in his *Universal Cambist*, an elaborate and useful work, states that the dram avoirdupois, like the drachm of the apothecaries, has sometimes been divided into 3 scruples and 60 grains; but as no such weight as an avoirdupois grain ever existed, the use of the expression is an instance of the confusion inseparable from having different systems of weights in which the same names are applied to things totally distinct.

Aside from the British statute weights, there are in England numerous other discordant denominations of weight, used for weighing different kinds of merchandise. One of the most common of these is the *stone*, which has a great variety of different significations. In London, however, only two stones are generally used, the one of 8 pounds for butchers' meat, and another of 14 pounds, for other commodities. A stone of glass has been reckoned at 5 pounds. The following are some of the other old weights: A *seam* of glass, equal to 24 stones; a *truss* of hay, equal to 56 pounds; a *truss* of new hay until the first of September, equal to 60 pounds; a *truss* of straw, equal to 36 pounds. In weighing wool the following denominations have been used: 7 pounds equal 1 *clove;* 2 cloves equal 1 *stone;* 2 stones equal 1 *tod;* 6½ tods equal 1 *wey;* 2 weys equal 1 *sack;* 12 sacks equal 1 *last;* a *pack* of wool equals 240 pounds.

In weighing cheese and butter, 8 pounds equal 1 clove, and 56 pounds equal 1 firkin. Many of these weights are now obsolete.

Ancient English Measures.—The basis of the ancient English measures of capacity and weight was the ancient Anglo-Saxon pound. This pound contained 5,400 grains, the grains being 22½ to the pennyweight, and the pennyweight being equal to 32 grains of average quality taken from the middle of the ear. Eight of these pounds formed the gallon of dry and liquid measure, 8 of these gallons the bushel, and 8 of these bushels the quarter. The old English pound stood to the Troy pound as 15 to 16. A few of these standard gallons and bushels have been preserved, and are found to be somewhat less than the proportion indicated. The Troy pound was not known as a legal standard until certain changes were introduced into the currency by Henry VIII.

The sack of wool was roughly reckoned as equal to the quarter of corn; 15 Saxon ounces formed the *libra mercatoria*, or pound of 7000 grains now called *avoirdupois*. Fourteen such pounds made the stone of wool, and 28 such stones constituted the sack. This calculation, however, makes the sack of wool lighter than the quarter of wheat by nearly four pounds.

Another ancient weight was the *charrus* of lead. It contained 2,100 avoirdupois pounds; and, divided by the old hundred, 108 pounds, is found to contain nearly 19½ hundred, which is the modern *fother* or *fodder*. The charrus contained 30 *fotmale*, or *pedes*, each pes containing 6 stone, less 2 pounds. The *foot* or *pig* of lead is the tenth of a cubic foot of lead. Iron was measured by the piece, 25 of which formed the hundred weight of 108 pounds. Wax and spices were estimated by the same hundred. A *last* of wool was 12 sacks; a last of herrings, ten thousand, each hundred being 120; a last of hides was 100, that is 10 dakers or dikers, each diker being ten.

THE AMERICAN SYSTEM.—Our system of weights was derived from those of the mother country. These, as already shown, were not framed by scientific men, but assumed their present form gradually, influenced by various circumstances, from which cause arose that irregularity which they exhibit. There are four kinds of weight in common use; Troy Weight, Apothecaries' Weight, Avoirdupois Weight, and Diamond Weight.

Troy Weight.—Troy Weight is used for weighing gold, silver, jewels, in ascertaining the strength of liquors, in philosophical experiments, etc. The term *Troy* is said to be derived from *Troyes*, the name of a town in France, where the weight was first used in Europe, it having been brought from Cairo in Egypt during the Crusades of the 12th century. Others maintain that it has no reference to any town in France, but rather to the monkish name given to London, of *Troy Novant*, founded on the legend of Brute, Troy weight being, therefore, London weight.

The term *pound* is from the Latin *pendo, I bend* or *weigh*. The term *ounce* is from the Latin *uncia*, a *twelfth* part, the ounce being one-twelfth of a pound. The *pennyweight* was the weight of the old English penny. The term *grain* is from a *grain* of *wheat*, which was the primitive standard of all the weights in England. Thirty-two of these taken from the middle of the ear and well dried, constituted a pennyweight, 20 pennyweights an ounce, and 12 ounces a pound. The pound thus derived was the legal standard of weight from the time of William the Conqueror to that of Henry VII. The latter king changed this weight and introduced the Troy pound, which was $\frac{1}{16}$ part, or $\frac{3}{4}$ of an ounce, heavier than the Saxon pound. The Troy pound was divided in the same manner as the Saxon pound, that is, into ounces, pennyweights, and grains; but the pennyweight contained only 24 grains, and consequently a grain Troy became much heavier than a grain of wheat.

The Troy pound is the *standard unit* of weight in the United States, and is the same as the Imperial pound Troy of Great Britain. It is equal to the weight of 22.794377 cubic inches of distilled water, at the temperature of 39.83° Fahrenheit, the barometer at 30 inches. In the United States Mint, the Troy ounce is adopted as the standard, and all weights are expressed in decimal multiples and sub-multiples of the ounce.

The symbol *oz.* is from the Spanish word *onza*, signifying ounce, though Webster derives it from the character ℨ, placed after the *O*, according to an ancient method of abbreviating terminations; *lb.* is from the Latin *libra*, a pound; *pwt.* is a combination of *p.* for penny, and *wt.* for weight; *dwt.*, from *denarius* and weight, is nearly obsolete, and seems a less appropriate term than pwt., being partly Latin and partly English.

Apothecaries' Weight.—Apothecaries' Weight is used only in mixing medicines. Apothecaries buy and sell their drugs by Avoirdupois Weight. The name arises from the weight being used by *apothecaries*. The term *scruple* is from the Latin *scrupulus*, a *little stone*. The term *dram* is from the Greek *drachma*, a *piece of money*.

The *symbols* have been supposed to be modifications of the figure 3, suggested by there being 3 scruples in a dram. Champollion, however, has traced them back to the hieroglyphics of Egypt. The unit is the pound, and is identical with the Troy pound, as are also the ounce and grain, the ounce being differently divided.

Avoirdupois Weight.—Avoirdupois Weight is used for weighing everything except jewels, gold, silver, liquors in philosophical experiments, etc. Avoirdupois Weight, it is said, was introduced by a statute of Henry VIII. The term occurs in some orders of his, A. D. 1532; and Queen Elizabeth, in 1588, ordered a pound of this weight to be deposited in the Exchequer as a standard. It was first used in England to weigh butchers' meat, but gradually came to be used to weigh all kinds of coarse goods or merchandise.

The term *Avoirdupois* is said to be derived from the French *avoir du poids*, signifying *to have weight.* Others think it is from *avoirs*, the ancient name of *goods* or *chattels*, and *poids*, signifying *weight* in the Norman dialect. Another authority says it is from the old French *aver de pes*, property or merchandise of weight, translated by Kelham, "any bulky commodities." Still another derivation is from the old French verb *averer*, to verify. The term *ton* is from the Saxon *tunne*, a *cask*. The origin of the other terms has already been given. The symbol *cwt* is from *centum* and *weight*.

The *unit* is the *pound*. It consists of 7000 Troy grains, and is consequently heavier than the pound *Troy*, which contains only 5,760. The ounce Avoirdupois, however, is lighter than the ounce Troy, owing to the difference in the division of the pound. A pound of feathers is therefore heavier than a pound of gold, while an ounce of gold is heavier than an ounce of lead.

The standard Avoirdupois pound of this country is the weight of 27.7015 cubic inches of distilled water, at its maximum density, or 39.83° Fahrenheit, weighed in the air, the barometer being at 30 inches. It is identical with the Imperial pound Avoirdupois of Great Britain, which is the weight of 27.7274 cubic inches of distilled water at the temperature of 62° Fahrenheit. The difference in the number of cubic inches is owing to the difference of temperature of the water employed.

In Great Britain 28 pounds equal 1 quarter, 112 pounds equal 1 hundredweight, and 2240 pounds equal 1 ton. These are called the *long hundred* and *long ton;* they were formerly used in this country, but are now only used at the custom-houses in invoices of English goods, in the wholesale iron and plate trade, and in wholesaling and freighting coal from the coal mines of Pennsylvania.

Diamond Weight.—Diamond Weight is used in weighing diamonds and other precious stones. In this weight, 16 parts equal 1 *grain*, and 4 *grains* equal 1 *carat*. One grain of this weight equals $\frac{4}{5}$ of a grain Troy. The term *carat* is also used to indicate a *proportional part* of a given weight, and is then called *assay carat*. Each assay carat consists of 4 *assay grains*, and each assay grain, of 4 *assay quarters*.

CHAPTER IV.

MEASURES OF VALUE.

THE Value of anything is its worth, or it is the property or properties of a thing which render it useful or estimable. It has also been defined as the estimate of a given commodity in comparison with other commodities. It is readily seen that it is not easy to give a definition of value entirely satisfactory. The two principal elements of the value of anything, are utility and difficulty of attainment.

Value is distinguished from price, which is the estimate given of any commodity by one value alone—the value of the precious metals. There can be a general rise in prices, since the value of the single measure may fall; but there cannot be a general rise in values, for values are relative and mutual. There may be, of course, a great rise or fall in any one thing when it is scarce and in demand, or abundant and neglected. There cannot be a universal rise or fall in values, for if such a case could be conceived, no one would get any more or less in either case than before.

The value of any commodity or service is affected by two causes—demand, which is temporary; and the cost of production, which is permanent. In the long run, a particular value conforms to the last named cause; but from time to time it is regulated by demand and supply, and may, therefore, rise and fall, far above or far below the cost of production. From the fact that demand and labor determine values, some economists have insisted that the first of these causes is the true measure of values; while others maintain that value is determined by the second. Both, however, are determining causes, though under different circumstances; and the statements, that labor is the cause of value, and demand is the cause of value, are, though apparently contradictory, two phases of the same fact.

A measure of value is one of the primal necessities of society. The united dependence of individuals creates the necessity of an exchange between two services or utilities. For such an act of exchange, it is essential that the things exchanged should be measured by some standard of value well understood between the contracting parties. To illustrate, suppose A produces shoes and B produces bread. Now A may want bread before B wants shoes. The immediate exchange of shoes for bread would not be convenient; but the exchange between the parties may still take place by the intervention of some medium of exchange. If B receives this medium for bread, he takes it in the faith that at his pleasure he may complete the exchange with A by the purchase of shoes, or may even employ the right to shoes assigned to him by the transfer of a portion of this medium, in procuring any other utility which he may desire. Hence a sale is said to be half of an exchange.

The necessity of some simple measure or representative of value, which could be employed as a medium of exchange, would be early felt. The objects first selected seem to have been organized bodies. Thus cattle, pigs, dried fish, and the skins of animals, have been used. The tendency, however, seems to have been to employ metallic substances. At first the baser metals were used as money. Iron was the primitive money of the Lacedemonians, and copper of the Romans. In most countries the precious metals were early adopted for this purpose. When first used, they were in the shape of bars or ingots, and were exchanged by weight. Aristotle and Pliny tell us that this is the method by which the precious metals were originally exchanged for other things in Greece and Italy. The Bible also states that Abraham weighed 200 shekels of silver and gave them in exchange for a piece of ground which he had purchased from the sons of Heth.

The medium of exchange, it is clear, must be, so far as possible, of persistent value, that is, liable to the fewest possible fluctuations of intrinsic value, and capable of being transferred for very nearly equal quantities of utilities at deferred periods. Hence the basis of a circulating medium must be a commodity which is of nearly absolute value. To possess this quality, it must be produced in nearly equal quantities by very nearly equal labor. Almost the only things which possess these characteristics are the precious metals, gold and silver; and consequently they have been almost universally selected as a medium of exchange. These two metals are still further adapted to a monetary use by being comparatively indestructible. Something is needed that can be treasured up for an indefinite period, without spontaneous alteration, waste, or decomposition, while waiting for an exchange. The material selected should also be homogeneous, or of equal value throughout its whole substance; and further, it must also be susceptible of easy division and reunion. Since these qualities are possessed almost exclusively by gold and silver, it is not surprising that all societies, spontaneously and as by instinct, have adopted the precious metals as money.

There is also a peculiar fitness found in the relation of gold and silver to each other, giving us two media of different fixed values. The circumstances under which they are found, and the labor required to produce them, are such that there has been but little disturbance in the mutual values of gold and silver; and although there can never be a precise ratio of intrinsic value possessed by each of the two metals, yet in modern times the margin of oscillation is so narrow, that both may be used simultaneously as media of exchange, the one for larger and the other for smaller values.

In order to answer for money, however, a mass of metal must be issued by an authority which gives to it a practical guaranty of its weight and fineness. It has thus been necessary for the government of every civilized country to coin its own money, to prevent the coinage by private parties, and to prohibit by severe penalties the forging of coins, the fabrication of counterfeit coins, or coins of less weight than the standard, or made up in whole or in part of some baser or less valuable metal. The necessity for this security is so great that however much governments have tampered with the weight of the nominal quantity, they have seldom, unless thoroughly demoralized and desperate, ventured on altering or debasing the standard; and when they have done so, the result has been ruinous in the last degree.

It was the practice in early ages to pay money by weight; from which it would seem that coins, in the strict sense of metallic masses of a certified weight, were unknown. The practice of weighing money, however, continued for ages after coins were in use. Where the system of coinage originated is not known, though it has been ascribed to different persons.

He who first shaped a metal into pieces of convenient size, marked with a distinct value, thus avoiding the need of the hammer and chisel to cut it off, and a balance to weigh it, was the first inventor of coins. History is silent respecting his name, his country, and the date of the invention. Homer speaks of workers in metals, but makes no mention of coined money. Herodotus says the Lydians, so far as he knew, were the first to use struck money, and there are reasons for thinking with him that the invention was Asiatic. The subject will be more fully considered subsequently.

Originally the coins of all countries seem to have had the same denominations as the weights commonly used in them, and contained the exact quantity of precious metals indicated by their name. Thus, the *talent* was a weight used in the earliest period by the Greeks, the *as* or *pondo* by the Romans, the *livre* by the French, and the *pound* by the English and Scotch; and the coins originally in use in Greece, Italy, France, and England, bore the same names and weighed precisely a *talent*, a *pondo*, a *livre*, and a *pound*. This arose from the original custom of making payments by the weight of the articles used as a medium of exchange. The standard has not, however, been preserved inviolate, in either ancient or modern times. It has been less degraded in England than anywhere else; but even there, the quantity of silver in a pound sterling is less than one-third part of a pound weight. In France, the livre current in 1789 contained less than one-sixty-sixth part of the silver implied in its name. In Spain and some other countries the depreciation has been carried still further.

When the use of coins has once been adopted, all values in contracts and other engagements are rated or estimated in money, and it is usual in almost all countries to enact that coins of legal and standard weight and purity shall be *legal tender*, and to declare that no legal proceedings of any kind shall be instituted on account of any debt or pecuniary obligation against any individual who has offered to liquidate the same by payment of an equivalent amount of the recognized coin of the country.

In the use of metals for money, it should be remembered there is simply an exchange of values. Equivalents are still given for equivalents. The exchange of a barrel of flour for an ounce of unfashioned gold is as much a barter as if it were exchanged for an ox or a barrel of beer; and if the metal were formed into a coin, or marked with a stamp declaring its weight and fineness, it would make no difference in the nature of the exchange. The notion has been entertained that coins are merely the signs of values. But they have no more claim to this designation than bars of iron or copper, sacks of wheat, or any other article. A draft or check may not improperly be regarded as a symbol of value; but a coin is itself an article of value. A dollar is not a sign; it is the thing signified.

The use of two precious metals, gold and silver, gives what is called a *double currency*. This produces some inconvenience on account of slight variations in the relative value of the two metals. In England, in 1803, the proportion was fixed at 1 to

15½, which slightly undervalued the customary proportion of gold, and hence gold was seldom or never seen. The great discoveries of gold in Australia and California slightly reversed the ratio, undervaluing the silver, the consequence of which is that gold has been, to a great extent, substituted for silver, and the latter metal exported in vast quantities. It seems likely, however, on account of improvements in methods of extracting silver from other ores and the cheapening of quicksilver, that the margin of oscillation will be considerably narrowed. Such oscillations, however, it is apparent, lead to many inconveniences in the use of a double currency, and France is said to be the only nation in which both metals are a legal tender. In England and our own country, gold is the actual standard. Silver and copper are issued, but are so much overvalued that they could not be exported in the shape of coins, and yet so regulated as to obviate the risk of private coining. In England, silver is not legal tender for more than 40s., nor copper for more than 12d., or if offered in farthings, for 6d. In the United States, silver is a legal tender up to $5. The German Empire has adopted gold alone as a legal tender; and Denmark, Sweden, the Netherlands, and some other countries, have done the same.

Paper Money.—But how great soever the advantages resulting from the employment of gold and silver as money, there are also many disadvantages. The use of a metallic currency is accompanied by a heavy expense; and there is a much greater difficulty in effecting payments by the use of coin than we might at first suppose. The cost of the wear and tear of the currency of a large country like England or the United States, allowing only a small percentage for the same, would amount to several millions of dollars a year. But the difficulty or inconvenience of the transportation of coin in making payments, is even a more serious objection. A million of dollars in gold would weigh nearly two tons, and would require a wagon to transport it. It is also inconvenient to make small payments between places remote from each other, inasmuch as the expense of sending gold by express, and the premium to guarantee it against loss, amount to a considerable sum. Hence arose the necessity of using for money some less valuable and more portable material than bullion; and hence, also, the origin of bills of exchange, checks, and other devices for economizing the use of money. The importance of such a representative currency appears also in the fact that without it at least four or five times as much gold and silver would be needed as with it. Indeed, without some such tokens or representatives of money, it would be almost impossible to meet the demands of barter and commerce in civilized countries.

Of the substitutes for gold and silver, *paper notes*, payable on demand, have been by far the most generally adopted, and are, in all respects, the most eligible. Intrinsically they are almost destitute of value, so that their employment and their loss costs next to nothing; and they may be carried about or transmitted by mail with the utmost facility. Possessing no value of themselves, their worth must depend entirely on artificial means or regulations. They are usually issued as substitutes for, or a representative of coin, the issuer being bound to pay in coin the sums they represent, on demand of the holder. For the issue of such notes it has been found necessary to establish banks of issue, and the questions that grow out of a banking system are among the most interesting that can present themselves to the mind of the economist. Bank notes, however, are not the only forms of a representative currency. Bankers' checks, private checks, bills of exchange, etc., and all analogous securities, accomplish the same purpose.

The basis of such securities, it must be remembered, is *confidence*—the confidence of the holder that they can be converted at his discretion into that which alone fulfills the conditions of a currency, the precious metals. It thus appears that the most of the business of civilized countries is not transacted with money, but upon *faith, confidence in promises,* a belief that men and institutions will do what they have promised. Notes are sometimes issued expressed in money value, but based on other values, as land, shares of stock, etc., and no harm ensues ordinarily, as no one will take them for more than they are worth; and since there is no compulsion, society will accept no more than it needs.

An essential element of such a currency is that it must be convertible and voluntary. Occasionally governments give their own paper, or the paper of some institution under their control, a forced circulation. Such an act disturbs the currency of a country, and often produces suffering and lasting injury to credit. A convertible currency cannot be extended beyond the amount which a community requires; but an inconvertible currency may be issued to any amount, and may be made to circulate extensively. The immediate effect of such an issue is to displace the metallic currency, which is either hoarded or exported. A very small premium on the precious metals is sufficient to banish them from circulation; and the forced currency being over-valued in comparison with the metallic currency, and of compulsory acceptance, no one will pay in the dearer medium; and the coin will command a premium, or paper will be at a discount.

Such an act on the part of a government is a wrong to the citizen as well as a source of disaster to the state. To promise money and give something else, is a fraud; and to force the acceptance of such a currency is as great a robbery of the public as the circulation of base money. The very fact of a circulation being compulsory is an indication that the currency is not, on its own merits, worth its nominal value; and the result is that such securities not only greatly depreciate, but often become worthless, and their repudiation inevitable. Among the most notable examples of such a currency are Law's bank under the Regency, the South Sea bubble, the French assignats, and, to a certain extent, the colonial currency during the American Revolution. It may be stated that governments seldom resort to such an act except in the exigencies of war; and it always requires years for the restoration of the proper relations between the paper and metallic currency.

History of Money.—Money has been employed as a medium of exchange from the very earliest historical periods. In ancient Greece and Rome, cattle were used, from which we derive our word pecuniary, which is from *pecunia*, and this from *pecus*, cattle. In early Greece there was a currency of "spits" or "skewers," six of which made a drachm, or handful; they were probably nails of iron or copper. The Lacedemonians and others used iron money. Among the most ancient existing specimens of coin are those of electrum, an alloy of gold with one-fifth of silver. Gold, silver, and copper were coined by the Greeks and Romans; tin was coined by Dionysius I., tyrant of Syracuse, and Roman and British tin coins are known to exist. Early leaden money is mentioned; a leaden stater is preserved in the British museum, and leaden money is now current in the Burman Empire. Numa Pompilius, King of Rome about 700 B. C., made money of both wood and leather. The Carthaginians had a kind of leather money; and the Emperor Frederick Barbarossa, 1158, and John the Good, King of France, 1360, also issued leather money. In 1574, when the city of Leyden was besieged by the Spaniards, leather money was used, and even quantities of pasteboard were coined in some parts of Holland. In the 13th century, money made out of the middle bark of the mulberry tree, cut into round pieces and stamped with the mark of the sovereign, was used in China. Cowry shells are used in Africa, in India, and the Indian islands, in the place of small coins. In India cakes of tea, in China pieces of silk, in Abyssinia salt, and in Iceland and Newfoundland codfish, pass for money. Wampum was used by the Indians; and about 1635 was the prevailing currency among the people of Massachusetts, became a legal tender, and was even counterfeited. About the same time corn and beans were used, and musket balls passed for change and were a legal tender for sums under one shilling. Notched wood was once used as a currency in England. So late as 1776, it was customary for workmen in Scotland to carry nails as money to the bake-shop and ale-house. It is thus seen that barter being one of the prime necessities of society, man finds, amid a variety of things, some one or more, according to circumstances, which will serve as the instrument of exchange.

The tendency, however, among all peoples, has been towards the precious metals, gold and silver. These metals, though first used by weight, were eventually coined into pieces of convenient and fixed values. The invention of coinage has been attributed to the wife of Midas, though without any historical certainty. High authority regards the Lydians, about 1200 B. C., as the inventors; and in support of this opinion it is also claimed that the earliest electrum coins, which undoubtedly belong to cities then under the dominion of the Lydian kings, seem to be of greater antiquity than any in the entire Greek series. By some Greek writers the invention is attributed to Phidon, King of Argos, in the 8th century B. C.; but he is now believed to have merely introduced coinage into Greece. It is said that the native bronze coins of China, the *tsien* or *cash*, bearing the inscription *tung-pan*, meaning current money, had its origin about 1200 B. C., at the beginning of the Chan dynasty. Lycurgus banished gold and silver and made the money of Sparta of iron, $100 worth of which required a cart and two oxen to draw it.

In Rome, for nearly 500 years after its foundation, no metal was coined but copper or brass. The *æs*, *as*, or *libra*, was a pound weight of copper or brass, stamped by the state, in the reign of Servius Tullius (578–534 B. C.). This coin, the unit of Roman money, was originally oblong like a brick, but subsequently was made round; and was cast, not struck. Before this reign, unstamped bars of copper were used for money Silver was first coined in Rome in 269 B. C., the principal coin being the *denarius;* and gold in 207 B. C., although it is held that the latter did not form a part of the regular currency until the time of Julius Cæsar. The Emperors possessed the privilege of coining gold and silver, but copper could be coined only by decree of the Senate.

The ancient Britons, at the invasion of Cæsar, had money of brass and iron, and it was paid by weight. During the reign of Augustus, one of the native kings caused money of gold, silver, and brass to be coined. Under the Emperor Claudius, the Roman money took the place of the Celtic, and continued in circulation until after the withdrawal of the Romans in the 5th century. The earliest coins subsequently issued are supposed to be the pennies of Ethelbert, King of Kent (560–616). These were coarsely stamped with the king's image on one side, and the name of either the mint-master or the city in which they were coined, on the other side. At this time, all money accounts began to be expressed in pounds, shillings, pence, and mancas or mancuses, although there was no coin but the penny, the other denominations being only moneys of account; 30 pence made a manca, 5 pence a shilling, and 40 shillings a pound. The mancas were reckoned both in gold and silver. In King Canute's laws the distinction is made that a *mancuse* was as much as a mark of silver, while a *manca* was a square piece of gold valued at 30 pence. King Athelstan (930) decreed that money should be uniform, and coined only in towns; and this decree mentions the fact that the clergy shared with the king the privilege of coining.

The Norman kings continued to coin only pence, which were of silver, and with a cross so deeply impressed that they might easily be broken into halfpence and farthings. The word *sterling*, to denote the standard money of England, is known to have been used as early as during the reign of William the Conqueror. Severe penalties were attached to the counterfeiting of money by Henry I. (1108), and during his reign halfpence were first coined. Henry II. (1154) found the money so much debased and reduced in value, that he provided for a new coinage, and punished those convicted of tampering with it. Silver farthings were first coined in 1222. In 1248 it was found that the money of the realm had been so clipped and otherwise defaced that its real worth bore no fixed proportion to its nominal value. Henry III. therefore ordered that the old coins should be brought to the mint and exchanged for new ones, weight for weight, thus entailing the entire loss, which was very great, upon the actual holders of these coins, which justly caused great complaint. During this reign, in 1257, gold pennies were first coined, which weighed $\frac{1}{120}$ of a pound tower, and passed for twenty pence. In 1279, Edward I. caused a new coinage of halfpence and farthings to be made, and provided that the old, which were principally mere fractions cut to suit, should no longer pass current. In 1300 he positively prohibited the circulation of any money not of his own coinage. In 1301 he diminished the weight of the pound sterling three pennies, equal to one per cent. Edward III. (1335), having exhausted his exchequer and embarrassed himself in his efforts to conquer France, ordered (1344) that in future 266 pennies should be made from a pound sterling; and two years subsequently he increased the number to 270 pennies. Shilling pieces were first coined during the reign of Henry VII., in 1505; and in 1523, during the reign of Henry VIII., silver farthings were coined for the last time. Queen Elizabeth raised the standard of silver coin, and in 1601 coined for Ireland shillings, sixpences, and threepences of a baser kind. The circulation of leaden tokens issued by the tradesmen of London was, to a great extent, stopped about the beginning of the 17th century. James I., in 1613, debased a portion of the coin, having coins in circulation of two qualities of fineness. James II. (1685–8) issued coins of tin, and authorized those of gun metal and of pewter. The first sovereigns were coined in 1489, under Henry VII.; half, quarter, and eighth sovereigns by Henry VIII., in 1544; and the first guinea by Charles II., in 1675.

Up to the 18th century, England had a double monetary standard, gold and silver; but owing to the over-valuation of silver in France, heavy silver coins disappeared from circulation, and the evil became so great that in 1774 it was declared that silver should no longer be a legal tender, except by weight, beyond £25. In 1816 the pound standard of silver was coined into 66*s*., the relative value with gold being as 1 to 14.287. Silver then became a legal tender for only 40 shillings. In 1792 Congress fixed the relative value of silver and gold at 1 to 15, which over-valuation of silver caused gold to be exported in such quantities that it was impossible to maintain a gold circulation. In 1834 the standard was changed to 1 to 16, while with other nations it was 1 to 15½, which caused silver to be so largely exported that in 1853 the ratio was changed to 1 to 14.88, and silver was made a legal tender only for sums under $5. By the coinage act of 1873, it was again changed to 1 to 14.95. For other interesting information upon this subject, see the American Cyclopedia, from which most of these facts have been taken.

Money.—That by which value is estimated is called *Money* Money may be defined as the measure or representative of the value of things. It is so called from the temple of Juno *Moneta*, in which money was first coined at Rome. It is of two kinds—*coin* and *paper money.* The money of a country is called its *currency*, from *curro*, I run, on account of its circulating through the country. The coin of a country is called its *specie currency*, and the paper money its *paper currency.* Coin is metal prepared to circulate as money. The metals used in this country are gold, silver, copper, and nickel. Paper money consists of printed promises to pay the bearer a certain amount, duly authorized to circulate as money.

United States Money.—The present system of our currency was established by an Act of Congress, August 8th, 1786. It was formerly, and is still sometimes, called Federal Money, because it was the money of the Federal Union. A plan for an American coinage was submitted to Congress in 1782, by Robert Morris, the head of the Finance Department, though its authorship is claimed for Gouverneur Morris. The plan adopted was that presented by Thomas Jefferson.

The standard unit of the system is the *dollar*, and the scale was made to conform to the decimal system of notation. The term *dollar* is probably of German origin, derived from *thal*, signifying a dale or valley. It is supposed that they were first coined about the year 1518, at Joachimsthal (Joachim's valley), a mining town of Bohemia, and called Joachims-thaler, and finally abbreviated to *thaler*. There are, however, several other theories to account for the word. Some German scholars derive the word *thaler* from *talent*, which was used, in the Middle Ages, to denote a pound of gold. Tooke says it is from the Anglo-Saxon *dael*, a portion, being a part or portion of a ducat. Thompson thinks that it is from the Swedish *daler*, from the town *Dale* or *Daleberg*, where it was coined. The dollar is a silver coin of Germany, Holland, Spain, Mexico, etc., though its value is not the same in all countries. In Spain, the coin is called *dalera*, the famous Spanish dollar, which for centuries figured so conspicuously in the commerce of the world. The Spanish dollar, called also the *milled dollar*, from its milled edge, was taken as the basis of United States coin and money account.

The term *dime*, one-tenth of a dollar, is derived from the French *disme*, meaning *ten;* the term *cent*, one hundredth of a dollar, from the Latin *centum*, a *hundred;* the term *mill*, one thousandth of a dollar, from the Latin *mille*, a *thousand*. The term *eagle* is probably applied on account of the design on the coin. The *cent* was proposed in 1782, by Robert Morris, and was named by Thomas Jefferson three years later. It was first coined in Vermont, in 1785, in the town of Rupert, and in the same year by Connecticut at New Haven, and by New Jersey and Massachusetts in 1786. The same year Congress authorized the establishment of a mint, but in 1787 they contracted with James Jarvis for 300 tons of cents, which were coined in New Haven. In 1792 a national mint was finally established under regulations which continued in force over forty years. The cent bore the head of Washington on one side and a chain of thirteen links on the other. The French Revolution creating a rage in America for French ideas, the image of Washington was deposed and the head of the Goddess of Liberty took its place, the chain being also replaced by the olive wreath of peace. French liberty was short-lived and so was the image upon the coin. The present face, with its classic features, was subsequently adopted, and has been but slightly changed since its adoption.

The origin of the symbol $, has never been satisfactorily determined. There are several theories, the most important of which are the following: 1st. It is supposed to be a combination of U. S, signifying United States, formed by writing the U over the S, which became changed in course of time to its present form. 2d. It is said to be a modification of the figure 8, denoting a piece of eight Reals or Testons. The dollar was formerly called "a piece of eight," and designated by the symbol $\frac{8}{8}$. 3d. It is said to be derived from the representation of the two "Pillars of Hercules," the ancient name of the opposite promontories at the Straits of Gibraltar. These were represented by two vertical lines connected with a scroll or label, and the coins containing this mark were called *pillar dollars* 4th. It is said to be a combination of HS., the mark of the Roman money unit. This symbol was prefixed to the numerals representing any sum, as the dollar mark is employed by us. The symbol HS. is a contraction of II., *two* and *Semis, half*, meaning two and a half; being equivalent to the word Sestertius, which was equal to two and one-half *nummi*. The Sesterce was the Roman money unit, as the dollar is ours. 5th. It is said to be a combination of P and S., from *peso duro*, or *peso fuerte*, meaning "hard dollar." In Spanish accounts this is always abbreviated by writing an S over a P, and placing the sign *after* the sum, as is also customary among the Portuguese.

The *coins* are of *gold*, *silver*, *bronze*, and *nickel* The gold coins are the *double eagle*, *eagle*, *half-eagle*, *quarter-eagle*, *three dollar*, and *one dollar*. *Fifty-dollar*, *half-dollar*, and *quarter-dollar* pieces are also coined, but are not legal circulation. The silver coins are the *dollar*, *half-dollar*, *quarter-dollar*, *dime*, *half-dime*, and *three-cent piece*. The bronze coins are the *two-cent* piece and the *cent*. The half-cent and cent of pure copper are not now coined. The mill has never been a coin, it is merely a convenient name for the tenth part of a cent. The nickel coins are the *five-cent* and *three-cent* pieces.

The gold and silver coins consist of nine parts pure metal and one part alloy, except the three-cent piece, which is one-fourth alloy. The alloy of the silver coin is pure copper; that of the gold coin is copper, or copper and silver, the silver not to exceed one-tenth of the whole alloy. The nickel coins contain one part nickel and three parts copper. The bronze coins contain 95 per cent. of copper and 5 per cent. of tin and zinc. The eagle weighs 258 grains, the other gold coins in proportion; the silver trade dollar, intended for commerce with China and Japan, weighs 420 grains; the half-dollar, 192.9 grains, and the other silver coins in proportion; the nickel five-cent piece weighs 5 grams (77. 16 grains nearly), and the three-cent piece weighs 30 grains; the bronze cent weighs 48 grains. The half-dollar, being half the weight of the five-franc piece of France, Belgium, and Switzerland, of the five-lire piece of Italy, the five-peseta coin of Spain, the five-drachma coin of Greece, and being equal in weight to the silver florin of Austria, is a step towards an international coinage.

Previous to the establishment of the decimal currency, we employed the currency of England, that is, pounds, shillings, and pence. Several of the States still use shillings and pence, though not with the same values. The difference in the number of shillings required for a dollar in the different States is owing to a depreciation in the paper money issued by the colonies. This depreciation was so great that in 1749, £1100 currency was only equal to £100 sterling. Soon after this Massachusetts receiving a remittance from England, called in her depreciated money at the rate of a Spanish dollar to 45 shillings of the paper currency, and the Legislature also passed an act fixing the par exchange between Massachusetts and England at £133⅓ currency to £100 sterling, and 6 shillings to the Spanish dollar. A similar depreciation of the paper money established the currencies of the other colonies. From this diversity in the colonial currencies it happens that the Spanish real of one-eighth of a dollar, was called in New England *ninepence;* in New York, *one shilling;* in Pennsylvania, *elevenpence* or a *levy.*

The earliest coinage in America was made in 1612, at the Somers Islands, now called Bermudas. The coin was of brass, with the legend "Sommer Island," and "a hogge on one side, in memory of the abundance of hogges which were found on their first landing." In 1645, the assembly of Virginia provided for a copper coinage, but the law was never executed The earliest colonial coinage was in Massachusetts, under an act passed May 27, 1652, which established a "mint howse" at Boston. The first coins were found to be too plain to prevent "washing and clipping," and were afterwards stamped with a figure of a tree, whence they were called "pine-tree shillings." In 1662 the assembly of Maryland passed an act "for the setting up of a mint within the province;" but it seems never to have been established. George I. attempted to introduce into the colonies, coins made of Bath metal, or pinchbeck; but this money was rejected by them. From 1778 to 1787, the power of coinage was exercised both by the confederation in Congress and by several of the individual states. The mint established in 1792 continued in operation under nearly the same regulations up to 1837, since which time numerous changes have been made, in both the value and the composition of the coins. In the United States, the right of coinage is vested by the Constitution in Congress, and prohibited to the several states; and yet individuals are left free to coin money, provided that the coins be not in "resemblance or similitude" of the gold or silver coins issued from the mint. Large amounts of private gold coins have been struck and circulated in different parts of the country. In the case of copper coins, however, the offering or receiving of any other copper coins than the cent and half-cent is prohibited by fine and forfeiture.

English Money.—English, or Sterling Money, is the legal currency of England. The *scale* is irregular, ascending by 4, 12, 20. The term *Sterling* is supposed to be derived from *Easterling*, the popular name of the Baltic and German traders who visited London in the Middle Ages. In what manner it came to be so applied is not certainly known. Camden says from the employment of German artists in coining. It is generally supposed that, these traders being called Easterlings, their money would naturally be called Easterling money, which was finally changed, by use, to Sterling money. The silver penny was first called Easterling. The unit is the pound, represented by the sovereign, and the £1 bank note.

The term *pound*, as a measure of value, is derived from pound, as a measure of weight, from the fact that anciently 240 pence were equal to a pound in weight, the term originally signifying a weight and not a value of money. The *penny* was formerly a silver piece, first coined by the Saxons. The term *farthing* is from *four things*. Previous to the time of Edward I., the penny was struck with a cross so deeply sunk in it, that it could be easily broken into halves and quarters, whence the names *half penny* and *four things* or *farthings*. Edward I. reduced the penny to a fixed standard, fixing its weight at the thirtieth part of an ounce. It afterward suffered successive diminutions until the reign of Elizabeth, when its value was fixed at the sixty-second part of an ounce of silver, which standard is still observed. The *shilling*, among the ancient Saxons, was only five pence. It subsequently underwent many alterations, containing sometimes 16 pence, and sometimes 20 pence. Its present value was fixed during the reign of Edward I. A coin of the same name is found in several other countries. The symbols £., s., d., qr., are the initials of the Latin words *libra*, *solidus*, *denarius*, and *quadrans;* signifying respectively pound, shilling, penny, and quarter. The old *ſ*, the original abbreviation for shillings, was formerly written between shillings and pence; thus 7s. 6d. was written 7*ſ*6. The *ſ* has since been changed into /, and shillings and pence are sometimes written thus, 7/6.

The English gold coins are the 5 *sovereign* piece, the *double-sovereign*, the *sovereign* and *half-sovereign*, the *guinea* and *half-guinea*. The Sovereign, equal to 20 shillings, represents the pound sterling. Its legal value in our currency is $4.8665. It is the standard gold coin. The Guinea, equal to 21 shillings, was first coined during the reign of Charles II., in the year 1662, of gold brought from Guinea, and hence its name. The guinea and half-guinea are no longer coined, though some of them are still in circulation. The silver coins are the *crown*, the *half-crown*, the *florin*, the *shilling*, the *sixpenny*, *four-penny*, and *threepenny* piece. The Crown is an old English coin, stamped with the figure of a crown, whence its name. Its value is 5 shillings sterling. The copper coins are the *penny*, *half-penny*, and *farthing*. The groat, worth 4 pence, is often mentioned.

The *noble*, the *angel*, and the *mark*, are old gold coins no longer in use. The Noble is an old coin of the Middle Ages, coined in the reign of Edward III. Its value is 6 shillings 8 pence. The Angel is an old coin valued at 10 shillings. It was impressed with the figure of an angel, in commemoration of a saying of Pope Gregory I., that the pagan Angli, or English, were so beautiful, that were they Christians, they would be angels. The Mark is an old coin, current in England and

Scotland, valued at 13 shillings 4 pence. A piece of money bearing this name, valued at 1 shilling 4 pence, is at present used in Hamburg.

The standard for gold coins is 22 carats fine, that is, 11 parts pure gold and one part alloy. This makes the English standard $\frac{1}{12}$ alloy, while the standard of the United States is $\frac{1}{10}$ alloy. The standard for silver is 37 parts pure silver and 3 parts alloy, hence the silver coins are $\frac{37}{40}$ pure and $\frac{3}{40}$ copper. Pence and half-pence are made of pure copper. The sovereign weighs 123.274 grains; the shilling weighs 87.27 grains; the penny weighs 240 grains, or $\frac{1}{2}$ ounce Troy.

The currency of Canada is the same as that of the United States, the table and denominations also being the same. The decimal currency was adopted in 1858, the Act taking effect in 1859, previous to which their currency was the same as the English. The coins consist of silver and copper. The *silver coins* are the *fifty-cent piece*, the *twenty-five-cent piece*, the *shilling* or *twenty-cent piece*, the *dime*, and the *half-dime*. The *copper coin* is the *cent*. The shilling equals about 19 cents of United States money; the values of the other silver coins are proportional. The silver coins consist of 37 parts silver to 3 of copper, the same as the English silver coins. There is no gold coinage, the British and American gold coins being a legal tender.

French Money.—The French system, like our own, is founded upon the decimal notation. The unit is the *franc*, whose value is fixed, by a late Act of Congress, at 19.3 cents. The franc is divided into tenths and hundredths, called respectively *decimes* and *centimes*. The *decime*, like our dime, is not used in business calculations, but is expressed by *centimes*. The gold and silver coins are nine-tenths pure metal.

German Money.—A new and uniform system of coinage has been adopted by the *New German Empire*. The unit is the *reichsmark*, worth 23.85 cents. A pound of gold, .9 pure, is divided into 139½ coins, and the tenth part of this coin is called a *mark*, and this is subdivided into 100 *pfennige*.

CHAPTER V.

MEASURES OF TIME.

TIME, in the sense here used, is regarded as a limited portion of duration, measured by certain conventional or natural periods. It is a definite portion of absolute Time, which is duration without beginning or end. The idea of Time in the absolute, is a grand intuition, like Space; Time, here considered, is known by experience and judgment.

Measures of Time were also originally derived from nature. The simplest unit of time, the *day*, "nature supplies ready made." The next simplest period, the *month*, or *mooneth*, is also presented to us by the changes of the moon constituting a lunation. For larger divisions than these, the phenomena of the seasons and the chief events from time to time occurring, have been used by early and uncivilized races. Among the Egyptians the rising of the Nile served as a point from which to reckon time. The New Zealanders were found to begin their year from the rising of the Pleiades above the sea. The migration of birds indicated the season to the Greeks and other nations. The Hottentot denoted periods by the number of moons before or after the ripening of his chief article of food. Barrow states that the Kaffir chronology is kept by the moon, and is registered by notches on sticks—the death of a favorite chief, or the gaining of a victory, serving for a new era. The peasantry of England refer to occurrences as "before sheep-shearing;" and in this country we often hear "harvest time," "after harvest," etc., used as dates of reckoning. It is, therefore, manifest that the more or less equal periods perceived in

Nature gave the first units of the measure of time, as in the cases already considered.

Historical.—As society progressed in civilization, the necessity of smaller and more precise units became apparent, which at last led to the adoption of some regular system adapted to the purposes of civil life. Such a system is called a Calendar, from the Latin *calare*, to call. In the early days of Rome it was the custom of the Pontiff to call the people together on the first day of each month, to apprise them of the days that were to be kept sacred during the month. Hence *dies calendæ*, the calends or first days of the different months.

The present calendars are derived from the Romans. Romulus is supposed to have first undertaken to divide the year in such a manner, that certain epochs should return periodically after the revolution of the sun; but the knowledge of astronomy was not then sufficiently advanced to allow this to be done with much precision. He placed the beginning of the year in the spring, and divided it into ten months,—March, April, May, June, Quintilis, Sextilis, September, October, November, and December. March, May, Quintilis, and October, contained 31 days each; the other six contained only thirty. The names Quintilis and Sextilis remained in the calendar till the end of the republic, when they were changed into July and August; the former in honor of Julius Cæsar, and the latter of Augustus.

The Roman month was divided into three periods by the *Calends*, the *Nones*, and the *Ides*. The *Calends* were invariably placed at the beginning of the month; the *Ides*, at the middle of the month, on the 13th or 15th; and the *Nones* (*novem*, nine) were the ninth day before the Ides, counting inclusively. From these three terms the days were counted backward in the following manner: those days comprised between the calends and the nones were denominated *days before the nones;* those between the nones and the ides, *days before the ides;* and those from the ides to the end of the month, *days before the calends.* Hence the phrases *pridie calendas, tertio calendas,* etc., meaning the second day before the calends, or last day of the month, the third day before the calends, or last but one of the month (the calends being included in the reckoning), and so on. In the months of March, May, July, and October, the ides fell on the 15th, and the nones, consequently, on the 7th. In all other months the ides fell on the 13th, and the nones, consequently, on the 5th. The number of days receiving their denomination from the calends, depended on the number of days in the month, and the day on which the ides fell. For example, if the month had thirty-one days and the ides fell on the 13th (as in January, August, and December), there would remain eighteen days after the ides, which, added to the first of the following month, made nineteen days of calends. Hence January 14th was styled the *nineteenth day before the calends* of February, and so on.

The year of Romulus, according to the mythical history, contained only 304 days. Numa, it is said, added two months; January to the beginning of the year, and February to the end. About the year 452 B. C. this arrangement was changed by the Decemvirs, who placed February after January; since that time the order of the months has remained undisturbed. In Numa's year the months consisted of 29 and 30 days alternately, to correspond with the synodic revolution of the moon. The year would therefore consist of 354 days; but one day was added to make the number odd, as being more lucky. In order to produce a correspondence with the solar year, Numa ordered an intercalary month to be inserted every second year between the 23d and 24th of February, consisting alternately of 22 and 23 days. Had this regulation been strictly adhered to, the mean length of the year would have been 365¼ days, and the months would have continued for a long time to correspond with the same seasons. But a discretionary power over the intercalary month was exercised by the pontiffs, for the purpose of hastening or retarding the days of election of magistrates; and thus the Roman calendar continued in a state of uncertainty and confusion till the time of Julius Cæsar, when the civil equinox differed from the astronomical by three months.

Under the advice of the astronomer Sosigenes, Cæsar abolished the lunar year, and regulated the civil year entirely by the sun. He decreed that the common year should consist of 365 days; but that every fourth year should contain 366. In distributing the days among the different months, he ordered that the odd months should contain each 31 days and the even months 30; excepting February, which in common years was to contain only 29 days, but every fourth year 30 days. This natural and convenient arrangement was interrupted to gratify the frivolous vanity of Augustus, by giving August, the month named after him, an equal number of days with July, which was named after Julius Cæsar. The intercalary day which occurred every fourth year was inserted between the 24th and 25th of February. According to the peculiar and awkward manner of reckoning adopted by the Romans, the 24th of February was called the 6th before the calends of March, *sexto calendas*. In the intercalary year this day was repeated, and called *bis-sexto calendas;* whence the term *bissextile*. The corresponding English term *leap year*, appears less correct, as it seems to imply that a day was *leapt* over instead of being thrust in. It may be remarked that in the ecclesiastical calendar, the intercalary day is still inserted between the 24th and 25th of February.

The Julian year consisted of 365¼ days, and consequently differed in excess by 11 minutes, 10.35 seconds from the true solar year, which consists of 365 days, 5 hours, 48 minutes, 49.7 seconds. In consequence of this difference the astronomical equinox, in the course of a few centuries, sensibly fell back towards the beginning of the year. In the time of Julius Cæsar it corresponded with the 25th of March; in the 16th century it had retrograded to the 11th. The correction of this error was one of the purposes sought to be obtained by the reformation of the calendar effected by Pope Gregory XIII.

in 1582. By suppressing 10 days in the calendar, Gregory restored the equinox to the 21st of March, the day on which it fell at the time of the Council of Nice, in 325. In order that the same inconvenience might be prevented in the future, he ordered the intercalation, which took place every fourth year, to be omitted in years ending centuries; excepting on the 400th, and the years which are multiples of 400. By this adjustment of the calendar the difference between a civil and a solar year will amount to a day in about 3,860 years.

The Gregorian calendar was received immediately or shortly after its promulgation by nearly all the Roman Catholic countries of Europe. The Protestant states of Germany and the kingdom of Denmark adhered to the Julian calendar until 1700; and in England the alteration was successfully opposed by popular prejudices until 1752. In that year the Julian calendar, or *old style*, as it was called, was formally abolished by the Act of Parliament, and the date used in all public transactions rendered coincident with that followed in all European countries, by enacting that the day following the 2d of September of the year 1752 should be called the 14th of that month. When the alteration was made by Gregory it was only necessary to drop 10 days; the year 1700 having intervened, which was a common year in the Gregorian but a leap year in the Julian calendar, it was now necessary to drop 11 days. The old style is still adhered to in Russia and the countries belonging to the communion of the Greek Church; the difference of date in the present century amounts to 12 days.

A new reform of the calendar was attempted in France during the period of the Revolution. The beginning of the year was fixed at the autumnal equinox, which nearly coincided with the foundation of the republic. The names of the ancient months were abolished, and others substituted having reference to agricultural labors, or the state of nature in different seasons of the year. But the change was found to be inconvenient and impracticable, and after a few years was formally abandoned.

Beginning of the Year.—There has been great diversity among different nations in the time of beginning the year. The ancient Egyptians, Chaldeans, Persians, Syrians, Phœnicians, Carthaginians, each began their year at the autumnal equinox, about the 22d of September. The Jews began their civil year at the same time, but their ecclesiastical year dated from the vernal equinox, about March 22d. Among the Greeks, until 432 B. C, when Meton introduced the cycle named after him, the year commenced at the winter solstice, about December 22d; and subsequently at the summer solstice, about the 22d of June. The Roman year from the time of Numa began at the winter solstice. Among the Latin Christian nations there were seven different dates for the beginning of the year: March 1; January 1; December 25; March 25 (beginning the year more than nine months sooner than we do, called the Pisan calculation); March 25 (beginning nearly three months later than we do, called the Florentine calculation); at Easter; and on January 1 (one year in advance of us). In France the year began in general at March 1, under the Merovingians; at December 25, under the Carlovingians; and at Easter under the Capetians. By edict of Charles IX., in 1564, the beginning of the year was ordered at January 1. In England, from the 14th century till the change of style in 1752, the legal and ecclesiastical year began at March 25, the day of the Annunciation, though it was not uncommon to reckon it in writing from January 1, the day of the circumcision. After the change, events which had occurred before March 25 in the old legal year, would by the new arrangement, be reckoned in the next subsequent year. Thus the revolution of 1688 occurred in February of that legal year, or, as we would now say, in February, 1689; and it was at one time customary to write the date thus,—February, $168\frac{8}{9}$.

The ancient northern nations of Europe began their year from the winter solstice. In the era of Constantinople, which was in use in the Byzantine Empire, and in Russia till the time of

Peter the Great, the civil year began with September 1, and the ecclesiastical sometimes with March 21, and sometimes with April 1. The beginning of the Mohammedan year, which is lunar, is not at any fixed time, but retrogrades through the different seasons of the solar year. The later Jewish year is lunar, but by the intercalation of a thirteenth month, seven times in a cycle of nineteen years, is brought into harmony with the solar periods; it begins at the autumnal equinox. Among most of the peoples of the East Indies the year is lunar, and begins with the first quarter of the moon the nearest to the beginning of December. Among the Peruvians, the year began at the winter solstice, and among the Mexicans at the vernal equinox. The year of the former was lunar, and was divided into four equal parts, bearing the names of their four principal festivals, instituted in honor of their four divinities allegorical of the sea sons. The Mexicans had a year of 360 days and 5 supplementary days. They divided it into 18 months of 20 days and had a leap year.

Lunar Year.—Though the return of the seasons obviously depends on the motion of the sun, or rather of the earth in its orbit, some nations have chosen to regulate their civil year by the motions of the moon; and many others have formed luni-solar years, by combining periods determined by the revolution of both bodies. The proper lunar year consists of twelve lunar months, or lunations, and consequently contains only 354 days; its commencement, therefore, anticipates that of the solar year by upwards of eleven days, and passes through the whole circle of the seasons in about 34 lunar years. The inconvenience attending this circumstance has been so universally perceived, that, excepting the modern Jews and Mohammedans, almost all nations which have regulated their months by the moon, have employed some method of intercalation for the purpose of retaining the beginning of the year at nearly the same place in the seasons. These methods are founded on certain luni-solar periods or cycles, which were established in the most ancient times, and which, with other relics of a barbarous age, are still preserved in our ecclesiastical calendars.

Ecclesiastical Calendars.—The adaptation of the civil to the solar year is attended with no difficulty; but the church calendar for regulating the movable feasts imposes conditions less easily satisfied. The early Christians borrowed a portion of their ritual from the Jews, whose year was luni-solar. Easter, the principal Christian festival, in imitation of the Jewish passover, was celebrated about the time of the full moon. Differences of opinion, and consequently disputations, soon arose as to the proper day on which the celebration should be held. In order to put an end to an unseemly contention, the Council of Nice laid down a specific rule that Easter should always be celebrated on the Sunday which immediately follows the full moon that happens upon, or next after, the day of the vernal equinox. In order to determine Easter according to this rule for any particular year, it is necessary to reconcile three periods; namely, the week, the lunar month, and the solar year. To find the day of the week on which any given day of the year falls, it is necessary to know on what day of the week the year began. In the Julian calendar this was easily found by means of a short period or cycle of 28 years, after which the year begins with the same day of the week. In the Gregorian calendar this order is interrupted by the omission of the intercalation in the last year of the century. But to render any calculation unnecessary, a table is given in the prayer-books, showing the correspondence of the days of the year and the week for the current century. The connection of the lunar month with the solar year is an ancient problem, for the resolution of which the Greeks invented cycles or periods, which remained in use with some modifications till the time of the Gregorian reformation. The author of the Gregorian calendar, Luigi Lilio Ghiraldi, or, as he is frequently called, Aloysius Lilius, employed for the same purpose a set of numbers called Epacts. It is to be desired that this complicated system of rules and tables were rendered unnecessary by abolishing the use of the lunar month, and causing Easter to fall invariably on the same Sunday of a calendar month; for example, the first or second Sunday in April.

The Measures.—Measures of Time are definite portions of duration, fixed by the revolutions of the earth on its axis and around the sun. The *unit* of time is the *day*, subdivided into *hours*, *minutes*, and *seconds*. The other divisions of time, arising from the *day*, are *weeks*, *months*, and *years*. The length of the day is determined by the revolution of the earth on its axis. The *Sidereal Day* is the exact time of the revolution of the earth on its axis. The *Solar Day* is the time of the apparent revolution of the sun around the earth. The *Astronomical Day* is the solar day, beginning and ending at noon. The *Civil Day* is the average length of all the solar days of the year; it begins at 12 o'clock midnight, and consists of two periods of 12 hours each.

The *second* and *minute* are parts of an hour, corresponding to the parts of a degree in Circular Measure. *Hour* is derived from the Latin *hora*, originally a definite space of time fixed by natural laws; a *day*, derived from the Saxon *daeg*, is the time of the revolution of the earth upon its axis; a *week* is a period of uncertain origin, but which has been used from time immemorial in Eastern countries; a *month*, from *monadh*, is the time of one revolution of the moon around the earth; a *year*, from Saxon *gear*, is the time of the earth's revolution around the sun; *century* comes from the Latin *centuria*, a collection of a hundred things.

The week is supposed, by some writers, to be derived from a tradition of the creation; by others, to have been suggested by the phases of the moon; while others refer its origin to the seven planets known in ancient times. This opinion explains the circumstance that the days of the week have been universally named after the planets, in a particular order. In the ancient Egyptian astronomy, the order of the "planets," in respect of distance from the earth, beginning with the most remote, was Saturn, Jupiter, Mars, the Sun, Venus, Mercury, the Moon. The day was divided into 24 hours, and each successive hour consecrated to a particular planet in the order just stated; and each day was named after the planet to which its first hour was consecrated. Supposing the first hour consecrated to Saturn, he would also have the 8th, 15th, and 22d hours. The 23d then would fall to Jupiter, the 24th to Mars, the 1st of the following day to the Sun, from which it would take its name. By proceeding in the same manner it is found that the third day would fall to the Moon, the fourth day to Mars, the fifth to Mercury, the sixth to Jupiter, the seventh to Venus, and, the cycle being completed, the eighth to Saturn again, and so continue in the same order. The Egyptians are said to have begun their week with Saturday. The Saxons seem to have borrowed the week from some Eastern nation, substituting the names of their own divinities for those of the gods of Greece

January is derived from *Janus*, the old Latin god of the sun and the year, to whom this month was held sacred. *February* is from *februa*, the Roman festival of propitiation, celebrated on the 15th of this month. January and February were added to the Roman calendar by Numa, Romulus having previously divided the year into 10 months. *March* is from *Mars*, the god of war and reputed father of Romulus. It was the first month of the Roman calendar. *April* is probably from the Latin *aperire*, to open, from the opening of the buds, or the bosom of the earth in producing vegetation. *May* is from *Maia*, the mother of Mercury, to whom the Romans offered sacrifices on the first day of this month. *June* is probably from *Juno*, the sister and wife of Jupiter. *July* was named by Mark Antony after Julius Cæsar, who was born in this month. It was previously called *Quintilis*. *August* was named after *Augustus* Cæsar. It was formerly called *Sextilis*, the sixth month. *September*, *October*, *November*, *December*, are respectively named from the Latin numerals, *septem*, *octo*, *novem*, *decem*.

Adjustment of the Calendar.—A True or Solar year is the exact time in which the earth revolves around the sun. It consists of 365 days 5 hours 48 minutes 49.7 seconds. Now, since it is inconvenient to reckon the fractional part of a day each year, it is necessary to arrange a correct calendar in which each year may have a whole number of days. This is done by causing some years to consist of 365 days and others of 366 days. The former are called common years; the latter, Bissextile or Leap years.

The calendar is reckoned according to the following rule:—*Every year that is divisible by* 4, *except the centennial, and every centennial year divisible by* 400, *is a leap year; all the other years are common years.* The centennial years are the hundredth years, or those which, when expressed in figures, end in two ciphers. The reason for this rule will now be explained.

If we reckon 365 days as 1 year, the time lost in the calendar in one year is 5 hours 48 minutes 49.7 seconds, and in 4 years is 23 hours 15 minutes 18.8 seconds, that is, *one day*, lacking only 44 minutes 41.2 seconds; hence the first error can be corrected by adding *one* day every *four* years, making the year to consist of 366 days.

If every fourth year be reckoned as leap year, since we add 44 minutes etc. too much, the time *gained* in the calendar in 4 years is 44 minutes 41.2 seconds, and in 100 years it will be 18 hours 37 minutes 10 seconds, that is, *one* day lacking 5 hours 22 minutes 50 seconds; hence the second error may be corrected by deducting one day from each centennial leap year, thus calling each centennial a common year of 365 days.

Again, if every centennial year be reckoned as a common year, since we do not add enough, the time lost in 100 years will be 5 hours 22 minutes 50 seconds, and in 400 years it will be 21 hours 31 minutes and 20 seconds; hence the time lost in

400 years will be 1 day lacking 2 hours 28 minutes 40 seconds, and this error may be rectified by making every fourth centennial year a leap year. In the same way we may make the calendar correct for any number of years.

The Gregorian method of intercalation thus gives 97 intercalations in 400 years; consequently 400 years contain 146097 days, and the mean length of the Gregorian year is 365 days 5 hours 49 minutes 12 seconds; exceeding the true solar year by 22.3 seconds; an error which amounts to one day in about 3866 years.

If an astronomer were required, without any reference to established usages, to give a rule for intercalation by which the civil year, while it always coincides with the commencement of a day, should deviate the least possible from the same instant of the solar year, he would proceed as follows: The fraction by which the solar year exceeds 365 days is .2422414, which, converted into a continued fraction, gives the following series of approximations,—$\frac{1}{4}$, $\frac{7}{29}$, $\frac{8}{33}$, $\frac{39}{161}$, $\frac{281}{1160}$, $\frac{320}{1321}$. Of these the first gives an intercalation of 1 day in 4 years, which supposes the year to be 365$\frac{1}{4}$ days. The second gives 7 intercalations in 29 years, and supposes the length of the year to be 365 days 5 hours 47 minutes 35 seconds, which is somewhat too small. The third fraction, $\frac{8}{33}$, is remarkable as giving a year which differs in excess from the true solar by 15.38 seconds, so that by intercalating 8 times in 33 years, or 7 times successively every fourth year, and once at the end of the fifth year, the difference between the civil and solar years would only accumulate to a day in about 5600 years; while in the Gregorian calendar the error amounts to a day in about 3866 years. The modern Persians are said, but not on very good authority, to intercalate in this manner. Nevertheless the Gregorian rule has the advantage that leap year is always readily distinguished.

Day of the Week.—There is a simple method of finding the day of the week upon which any year begins, and also the day of the week upon which any event has occurred, which we will explain. Take the first seven letters of the alphabet, A, B, C, D, E, F, G, A representing the 1st of January, B the 2d, C the 3d, etc., A again the 8th, B the 9th, and so on through the year. Now, one of these letters will stand for Sunday, and this letter is called the *Sunday* or *Dominical Letter*. Thus, if January begins on Sunday, A will be the dominical letter for that year; if January begins on Monday, the first Sunday will be the 7th; hence G, the 7th letter, will be the dominical letter. In leap year we have two dominical letters, one for January and February, and the next preceding for the remainder of the year.

We find the dominical letter for any year according to the Julian, or Old Style, by the following rule: *To the given year add one-fourth of itself, plus* 4, *and divide the sum by* 7. *If there is no remainder, the dominical letter is G; if* 1 *remains, F; and so on in reverse order.* In leap year the letter thus found will be the dominical letter for the last ten months, and the next following letter for the first two months. Having the dominical letter, we can easily find the day on which the year begins.

When the year is reckoned according to the New Style, we find the dominical letter by the following rule: *Divide the number of the century by* 4, *subtract the remainder from* 3, *add twice this remainder to the odd years plus* $\frac{1}{4}$ *of the odd years, and divide the sum by* 7. *If there is no remainder, the dominical letter is G; if* 1 *remains it is F, etc.*

Having found the day of the week on which January begins in any year, we may find on what day each month commences by the following couplet:

At Dover Dwelt George Brown Esquire,
Good Captain French and David Friar.

The initial letters of the several words indicate the months in their order, the first word the first month, the second word the second month, etc. Now, if January begins on Sunday, D, the letter for February, being the fourth of the series, indicates that February begins on the fourth day of the week, or Wednesday, etc. In leap year, the months after February come in one day later.

The method may be illustrated by the following example: Upon what day of the week did the battle of Bunker Hill occur, June 17, 1775? Dividing 17, the number of the century, by 4, we have a remainder of 1; subtracting this from 3 and multiplying by 2 we have 4; 4 plus 75, plus $\frac{1}{4}$ of 75, omitting the fraction, gives 97, which divided by 7 gives a remainder of 6; hence the dominical letter is 6 before G, or A. Therefore the 1st day of January was Sunday. Now, by the couplet, the first day of June is E; and as A is the dominical letter, June 1st was on Thursday, and the 17th was on Saturday.

In closing this chapter, we call attention to a simple point upon which there has been some popular misapprehension — that is, the manner of reckoning the centuries. The centuries are numbered from the beginning of the Christian era. Thus, all events transpiring from the beginning of this era until the end of the first hundred years are reckoned as belonging to the first century; all events between the end of the 100th year and the end of the 200th year, are reckoned in the second century, etc. It is thus seen that the 18th century closed with the end of the year 1800, and the 19th century commenced at the beginning of the year 1801.

CHAPTER VI.

THE METRIC SYSTEM.

THE Old System of weights and measures was born of necessity and developed by chance. Some method of measuring quantity must have been coeval with the race, being a necessity of man as a social being; and the earliest systems were gradually enlarged and improved as man advanced in civilization. Originating by chance or circumstances, however, rather than by science, they lacked symmetry and precision, were difficult to learn and inconvenient to apply.

The two principal characteristics of a system of weights and measures are the *units* and their *scale of relations*. The essentials are that the units should be *precise* and the scale *regular*. Neither of these conditions exists in the old system. The earliest of measures were derived from variable objects in nature, and the scales of relations are so capricious as to defy an attempt to account for their origin. No two tables have the same scale, and seldom have more than two units of a scale the same relation.

The units of the old system derived from natural objects, and without any scientific considerations. were necessarily uncertain. The *foot*, among different nations, would vary even more than the size of their real feet, and the same lack of precision would belong to all other measures. But the scale of relations, arising from accidental circumstances, and being controlled by no scientific principle, was, if possible, even more irregular. About thirty different numbers are used in the ordinary tables of the United States and England; and these are arranged without any regard to convenience or law. Such is the system of weights and measures that we have been required to learn and apply to the business transactions of life.

Science found the old system in the condition described, and endeavored to reform it. It established the units upon scientific principles, and thus gave it precision. The yard, which at one time was the length of the king's arm, was fixed by the vibrations of the pendulum, and from it all other units of measure and weight were derived. But though science could give exactness to the old system, it could not impart to it simplicity and regularity. The confusion was too great even for the touch of science to reduce to order. There was only one way in which such a reform could be accomplished, and that was to throw away the old system and construct a new one. This was done, and the result is the Metric System.

History.—This system was suggested as long ago as 1528, by Jean Fernal, a physician of Henry II., of France; but the suggestion did not take a practical turn until 1790, when Prince Talleyrand distributed among the members of the Constituent Assembly of France a proposal, founded upon the excessive diversity and confusion of the weights and measures then prevailing all over that country, for the formation of a new system upon the principle of a single and universal standard.

A committee of the Academy of Sciences, consisting of five of the most eminent mathematicians of Europe,—Borda, Lagrange, Laplace, Monge, and Condorcet,—was subsequently appointed, under a decree of the Constituent Assembly, to report upon the selection of a natural standard; and the committee proposed in its report that the ten-millionth part of the quarter of the meridian of Paris should be taken as the standard unit of lineal measure.

Delambre and Méchain were appointed to measure an arc of the meridian between Dunkirk and Barcelona, as Cassini had been appointed in 1669. They commenced their labors at the most agitated period of the French Revolution. At every station of their progress in the field-survey, they were arrested by the suspicions and alarms of the people, who took them for spies or engineers of the invading enemies of France. The result was a wonderful approximation to the true length, and one in the highest degree "creditable to the French astronomers and geometricians, who carried on their operations under every difficulty and at the hazard of their lives, in the midst of the greatest political convulsion of modern times."

The arc of the meridian extending from Dunkirk to Barcelona, comprising about 10° of latitude, was measured trigonometrically and compared with the arc measured by Bouguer and La Condamine in Peru in 1736; and the length of the quarter of the meridian, or the distance from the Equator to the Pole, was calculated. This length was divided into ten million equal parts, and one of these parts was taken for the unit of length and called a *mètre*, from the Greek word *μέτρον*, a *measure*. The distance was measured in terms of the *toise*, or old fathom (six-foot) measure of France, which was used as the measurement of the base lines; and its ten-millionth part, or the Mètre, was determined to be 443.296 lines, the line being $\frac{1}{144}$ of a foot. It appears thus that four mètres would exceed two toises by the 19th part of a toise, very nearly; and the following process of constructing the mètre was adopted. Nineteen pieces were made as nearly as possible equal to each other, so that their aggregate would be a toise; upon examination it was found that one had almost exactly the required length. This piece, together with the two toises that had served for the base measurements, was placed in the comparator and compared with four single mètre bars abutted together, which were similarly compared with each other, and adjusted by grinding and polishing their ends until they had the desired length. These bars were, like the toises, of iron; one of them was chosen for the French standard, from which the platinum mètre of the archives, which is the legal standard of France, was copied. Another of these original mètres was brought to the United States and has served as the standard for the geodesy of the coast survey, and for the construction of a metric standard for this country.

If the arc of the meridian is calculated from the result of French researches, the mètre itself is equal, in English measurement, to 39.37079 inches; and multiplying this length by ten millions, the length of the quadrant of the meridian when converted into feet, will be 32,808,992 feet. Sir John Herschel estimates the length of the quadrant of the meridian at 32,813,000 feet; so that, according to his calculation, there is a difference between the French and the new estimate of the quadrant of 4008 feet, and therefore the French length of the quadrant is $\frac{1}{8194}$ too short, and the mètre is $\frac{1}{208}$ of an inch less than the length of the ten-millionth part of the quadrant.

This error of $\frac{1}{208}$ of an inch in the determination of the mètre, however, supposing it possible to establish it absolutely, does not make the metric system less complete or useful; but is more than counterbalanced by the extreme simplicity, symmetry, and convenience of the system. Professor Bessel observed with respect to the mètre, that in the measurement of a length between two points on the surface of the earth, there is no advantage at all in proving the relation of the measured distance to a quadrant of the meridian. Professor Miller, of Cambridge, also deems the error in the relation of the mètre to the quadrant of the meridian to be of no consequence. An error which has also been discovered in the kilogramme is pronounced by the same authorities to be of little importance in a practical point of view.

It is difficult to make an exact comparison between the mètre and the inch or yard, arising from the fact that the mètre is an end measure of platinum, having its standard length at 32° F., while the yard is a line measure of bronze, standard at 62° F. They cannot, therefore, be directly compared, and the dilatation by temperature comes into effect, and requires to be ascertained with the utmost accuracy. The means of comparison for standards of length are different according to their being line or end measures. In the former case, when, as in the British yard, the standard length is contained between lines drawn upon the bar, the comparator is necessarily optical, which enables us to measure by means of micrometer microscopes the minute differences between different measures traced from the same standard by mechanical means. But when the standards are end measures, or contained between the terminal planes of the bar, the comparison is necessarily made by actual contact, the rotation of a mirror, or tilting of a delicate level, being used as the means of indicating the minute differences. In standard measures of the latter kind, it is now usual to make the terminal surfaces very small, and ground off parallel to each other by means of cylindrical bearings near each end of the bar. It is only by such means that parallelism approaching to geometrical accuracy can be obtained. In both kinds of comparison, a precision of the $\frac{1}{100000}$ part of an inch may be reached. The greatest difficulty in obtaining extreme precision arises from the variability of temperature; and this is greatly enhanced when the measures compared are of different volume, and still more when of different metals. In comparisons of precision, it is therefore necessary, to insure a great uniformity of temperature, to prevent as much as possible the influence of the bodily heat of the observer upon the apparatus.

The System.—The Metric System is so called because the *base*, or fundamental unit from which all the other units are derived, is the *Meter*. The units of the different measures have simple and definite relations, and the whole system is founded upon the decimal scale. A unit of any measure being established, the other denominations are derived by taking decimal multiples and divisions of the unit. The multiples are named by prefixes derived from the Greek,—*deka*, ten; *hecto*, hundred, etc.; those denoting divisions, from the Latin,—*deci*, tenth, etc. The scale ascends and descends by tens, the same as our ordinary scale of numeration and notation. Any quantity consisting of several denominations is thus written and treated like an integer and decimal, the decimal point separating the unit and its divisions. Another merit of the system is that the units are all correlated, being derived from the standard unit or *Meter*.

The *Unit of Length* is the *Meter*. It equals 39.37 inches, which is in theory the ten-millionth part of a degree of latitude, being a little longer than the yard, the present unit of length. Another expression for its length which may be easily remembered, is 3 feet, 3 inches, and 3 eighths of an inch. The *Meter* is the unit from which all the other units are derived. It is the basis, the fundamental unit, of the whole system. Ordinary lengths are measured by the *meter;* very small distances by the *millimeter* ($\frac{1}{1000}$ of a meter), and long distances by the *kilometer* (1000 meters). The five-cent piece, adopted in 1866, is $\frac{1}{50}$ of a meter or 2 centimeters in diameter. A *decimeter* is about 4 inches; a *millimeter*, about $\frac{1}{25}$ of an inch; a *kilometer*, about 200 rods, or $\frac{5}{8}$ of a mile.

The *Unit of Surface*, used in measuring land, is the *Are*, which is a *square decameter*. It equals 3.9574 perches, or 0.0247 acre. The *are*, *centiare*, and *hectare* are the denominations principally used, as they are exact squares; the *deciare* is not a square, but merely the tenth of an are, and the *decare* is merely ten ares. The *centiare* is a square meter. A *hectare* is nearly $2\frac{1}{2}$ acres; an acre is nearly 40 ares. Other surfaces, as cloth, lumber, paper, etc., are measured by the *square meter*.

The *Unit of Volume* used in measuring wood, is the *Stere*, which is a *cubic meter*. The *stere*, *decistere*, and *decastere* are principally used. 3.6 *steres* very nearly equal a cord. Other solid bodies, like stone, sand, gravel, etc., are measured by the *cubic meter* and its divisions.

The *Unit of Capacity* is the *Liter*, which equals a *cubic decimeter*, and contains 61.027 cubic inches, or 2.1135 pints wine measure, or 1.816 pints dry measure. This measure is used both for measuring liquids and dry subtances. Liquids are usually measured by the *liter*, and grain by the *hectoliter*, which equals nearly 2$\frac{5}{6}$ bushels, or $\frac{5}{6}$ of a barrel. 4 *liters* are a little more than a *gallon*, and 35 *liters* nearly a *bushel*.

The *Unit of Weight* is the *Gram*. It is the weight of a *cubic centimeter* of distilled water at the temperature of melting ice, and equals 15.432 Troy grains. It is used in weighing letters, in mixing and compounding medicines, and in weighing all very light articles. The five-cent coin adopted in 1866, weighs 5 grams. The *kilogram*, usually abbreviated into *kilo*, is the ordinary unit of weight. It equals about 2$\frac{1}{5}$ pounds Avoirdupois. Meat, sugar, etc., are bought and sold by the kilogram. In weighing heavy articles, two other weights, the *quintal* (100 kilograms) and the *tonneau* (1000 kilograms) are used. The U. S. Post-office receives 15 grams, though a little over weight, as equivalent to an ounce Avoirdupois.

Its Adoption.—The Metric System was adopted in France in 1795, but was tardily accepted by the people. Though it was early made compulsory, it became necessary to relax the law so as to permit the use of halves and quarters of the several units. Since 1840, however, the metric measures have been the only ones in common use in France; and the system has found a very large acceptance elsewhere. It has been adopted by Italy, Spain, Portugal, Greece, Belgium, Holland, and partially by Denmark and Switzerland. Many of the German states have also expressed their approval of the system, and the half kilogram has been introduced into all the great mercantile operations in Austria. In 1863, at the International Statistical Congress held at Berlin, a resolution was passed recommending the metric system as the most convenient for international measures. A Commission of the Imperial Academy of Sciences in St. Petersburg has recommended that such alterations be made in Russian weights and measures as would put them in conformity with the French system.

In England, the metric system has been extensively used among scientific men for many years; and in 1864 the Parliament passed an act legalizing its use throughout the British Empire. In 1866, the Congress of the United States authorized its use in this country; and to facilitate its introduction directed that the new five-cent piece should weigh *five grams* and be *one-fiftieth* of a meter in diameter. Owing to the composition of the alloy, it was found necessary to make its diameter a little greater than *one-fiftieth* of a meter; 48.6 nickel five-cent pieces, laid side by side, measure one meter. It was also ordered that letters in post-offices should be weighed by the gram, but this latter provision has not been carried out. There is, however, a strong feeling among scientific men in favor of the system; and the time is not very far distant when it will be universally used in this country. And more than this,—to the honor of France and the scientific men who suggested and perfected it,—the metric system is destined to become the system of the civilized world.

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APPENDIX.

HENKLE'S NAMES OF PERIODS.

Millions (1), Billions (2), Trillions (3), Quadrillions (4), Quintillions (5), Sextillions (6), Septillions (7), Octillions (8), Nonillions (9), Decillions (10), Undecillions (11), Duodecillions (12), Tertio-decillions (13), Quarto-decillions (14), Quinto-decillions (15), Sexto-decillions (16), Octo-decillions (18), Nono-decillions (19), Vigillions (20), Primo-vigillions (21), Secundo-vigillions (22), Tertio-vigillions (23), Quarto-vigillions (24), Quinto-vigillions (25), Sexto-vigillions (26), Septo-vigillions (27), Octo-vigillions (28), Nono-vigillions (29), Trigillions (30), Quadragillions (40), Quinquagillions, (50), Sexagillions (60), Septuagillions (70), Octogillions (80), Nonagillions (90), Centillions (100), Primo-centillions (101), Decimo-centillions (110), Undecimo-centillions (111), Duodecimo-centillions (112), Tertio-decimo-centillions (113 , Quarto-decimo-centillions (114), Vigesimo-centillions (120), Primo-vigesimo-centillions (121), Trigesimo-centillions (130), Quadragesimo-centillions (140), Quinquagesimo-centillions (150), Sexagesimo-centillions (160), Septuagesimo-centillions (170), Octogesimo-centillions (180), Nonagesimo-centillions (190), Ducentillions (200), Trecentillions (300), Quadringentillions (400), Quingentillions (500), Sexcentillions (600), Septingentillions (700), Octingentillions (800), Nongentillions (900), Millillions (1000), Centesimo-millillions (1100), Ducentesimo-millillions (1200), Trecentesimo-millillions (1300), Quadringentesimo-millillions (1400), Quingentesimo-millillions (1500), Sexcentesimo-millillions (1600), Septingentesimo-millillions (1700), Octingentesimo-millillions (1800), Nongentesimo-millillions (1900), Bi-millillions (2000), Tri-millillions (3000), Quadri-millillions (4000), Quinqui-millillions (5000), Sexi-millillions (6000) Septi-millillions (7000), Octi-millillions (8000), Novi-millillions (9000), Deci-millillions (10,000), Undeci-millillions (11,000), Duodeci-millillions (12,000), Tredeci-millillions (13,000), Quatuordeci-millillions (14,000), Quindeci-millillions (15,000), Sexdeci-millillions (16,000), Septi-deci-millillions (17,000), Octi-deci-millillions (18,000), Novi-deci-millillions (19,000), Vici-millillions (20,000), Semeli-vici-millillions (21,000), Bi-vici-millillions (22,000), Tri-vici-millillions (23,000), Quadri-vici-millillions (24,000), Quinqui-vici-millillions (25,000), Sexi-vici-millillions (26,000), Septi-vici-millillions (27,000), Octi-vici-millillions (28,000), Novi-vici-millillions (29,000), Trici-millillions (30,000), Quadragi-millillions (40,000), Quinquagi-millillions (50,000), Sexagi-millillions (60,000), Septuagi-millillions (70,000), Octogi-millillions (80,000), Nonagi-millillions (90,000). Centi-millillions (100,000), Semeli-centi-millillions (101,000), Bi-centi-millillions (102,000), Ducenti-millillions (200,000), Trecenti-millillions (300,000), Quadringenti-millillions (400,000), Quingenti-millillions (500,000), Sexcenti-millillions (600,000), Septingenti-millillions (700,000), Octingenti-millillions (800,000), Nongenti-millillions (900,000), Milli-millillions (1,000,000).

It should be observed that words ending in *o* represent numbers to be added, and those ending in *i* represent multipliers. When two words end in *i*, the sum of the numbers indicated is to be taken as the multiplier. In each, the last word indicates the number to be increased or multiplied.